Time-local Formulation of Passive Impedance Boundary Conditions

Workshop on Herglotz-Nevanlinna functions, CIRM 25th May 2022

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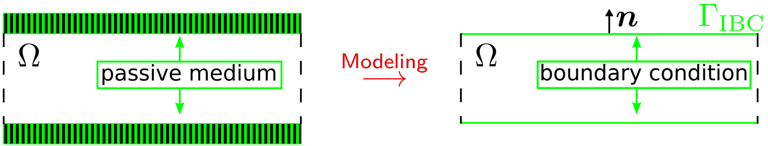
Contents

What is an impedance boundary condition (IBC)?

Purpose: model a passive medium as a boundary condition.

Typical application: waveguide



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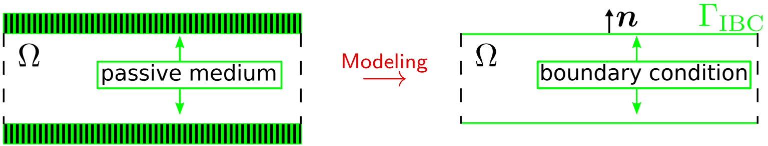

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$$\partial_t p(\boldsymbol{x}, t) = -z \star \partial_n p(\boldsymbol{x}, t) \qquad (\boldsymbol{x} \in \Gamma_{\text{IBC}})$$

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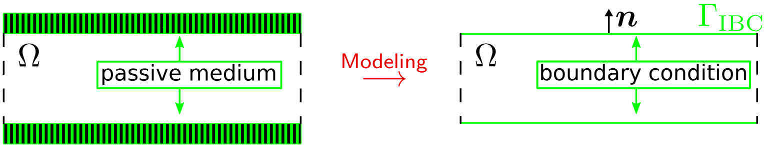

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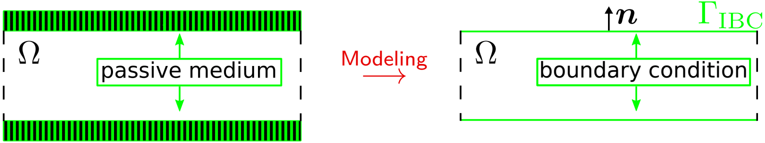

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Example 2: Maxwell's equations

$$\boldsymbol{E}_{||} = z \star \boldsymbol{H}_{||} \times \boldsymbol{n} \quad (\boldsymbol{x} \in \Gamma_{\mathsf{IBC}})$$

Can we always use an IBC?

Accurate if:

- homogenization is possible
- homogenized medium is highly anisotropic

Example: Helmholtz resonator ("acoustic liner")

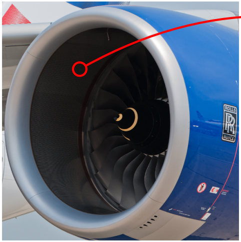

Fig. Trent 900 (A380). Inlet lined with a sound absorbing material.

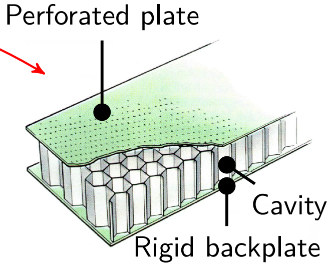

Fig. Example of liner.

Extensions:

- nonlinear absorption $\rightarrow \partial_t p = -\mathcal{Z}\left(\partial_n p\right)$
- flow effect (Joubert 2010) (Khamis and Brambley 2017)

Outline

Talk objective. Overview of impedance boundary conditions (IBC):

- Derive numerical models from physical models
- Formulation suited to numerical methods for hyperbolic laws

Outline

2. Admissibility conditions
   What is the class of admissible impedance operators?
3. Extended formulations
   What is the structure of physical impedance models?
4. Numerical applications
   How to efficiently discretize an IBC?

Contents

"System theory" viewpoint

Let us consider an IBC given by an operator $\mathcal{Z}$.

Power balance along trajectories

Wave equation:

$$\mathcal{E} := \frac{1}{2}\|\partial_t p\|_\Omega^2 + \frac{1}{2}\|\nabla p\|_\Omega^2$$

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\Re \int_{\partial\Omega} \mathcal{Z}(\partial_n p)\,\overline{\partial_n p} \overset{?}{\leq} 0$$

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⚠ "System theory" viewpoint. Forget about the PDE and study

$$\mathcal{Z} : u(t) \mapsto y(t)$$

as an operator acting on functions of time.

A. Zemanian (1965). *Distribution Theory and Transform Analysis*. New York: McGraw-Hill

E. J. Beltrami and M. R. Wohlers (1966). *Distributions and the boundary values of analytic functions*. New York: Academic Press

Next: admissibility conditions on $\mathcal{Z}$.

Admissibility conditions: summary

Starting point: let $\mathcal{Z}$ be a continuous map $\mathcal{E}' \to \mathcal{D}'$.

Definition. $\mathcal{Z}$ is *passive* if $\forall u \in \mathcal{C}_0^\infty(\mathbb{R})$, $\forall t > 0$,

$$\mathcal{E}(t) := \int_{-\infty}^{t} \mathcal{P}(\tau)\,\mathrm{d}\tau \geq 0, \quad \text{where} \quad \mathcal{P}(\tau) := \Re[\mathcal{Z}(u)(\tau)\overline{u(\tau)}].$$

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Theorem. $\mathcal{Z}$ is linear time-invariant and admissible
$\Leftrightarrow \mathcal{Z}(u) = z \star u$ with $z \in \mathcal{D}'_+ \cap \mathcal{S}'$ and $\hat{z}$ **positive-real**.

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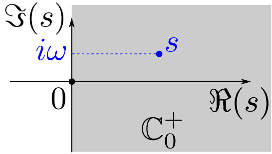
Definition. $f : \mathbb{C}_0^+ \to \mathbb{C}$ is ***positive-real*** if
(i) f is analytic,
(ii) $\Re[f] \geq 0$,
(iii) $f(s) \in \mathbb{R}$ when $s \in (0, \infty)$.



Lemma. f satisfies (i) and (ii) $\iff$ $z \mapsto i\,f\left(\frac{z}{i}\right)$ Herglotz.

Contents

Objective: getting rid of the convolution!

Wave equation with linear time-invariant IBC:

$$\partial_t \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = - \begin{pmatrix} \nabla p \\ \nabla \cdot \boldsymbol{u} \end{pmatrix} \quad \text{on } \Omega, \quad p = z \star \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega.$$

⚠ Difficulty: we need $z \star u(t)$ but we only know $\hat{z}(s)$.

Extended formulation. Abstract Cauchy problem:

$$\partial_t X = \mathcal{A} X, \quad X = (p, u, \varphi) \in \mathcal{H},$$

with $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$.

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Example. ODE realization of z:

$$z \star \boldsymbol{u}(t) \cdot \boldsymbol{n} = \int_\Gamma \varphi_s(t) \, \mathrm{d}\mu(s)$$

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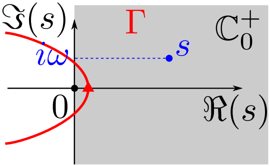

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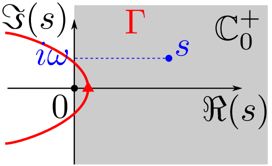

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Two formulations: 1. conservative. 2. dissipative.

Conservative realization ($\Gamma = i\mathbb{R}$)

Representation. If $\hat{z}$ is positive-real, (Nedic 2017) (Cassier and Milton 2017)

$$\hat{z}(s) = a\,s + \int_0^\infty \frac{s}{\omega^2 + s^2}\,\mathrm{d}\nu(\omega) \quad (\Re(s) > 0)\,,$$

with $\mathrm{d}\nu(\omega) = \frac{2}{\pi}\Re[\hat{z}(i\omega)]\,\mathrm{d}\omega,\ a = \lim_{x\to+\infty}\frac{\hat{z}(x)}{x} \geq 0.$

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Realization. There is a skew-symmetric matrix J_ω such that

$$z \star u = a\partial_t u + \int_0^\infty \varphi_\omega\,\mathrm{d}\nu(\omega),\ \partial_t \begin{bmatrix}\varphi_\omega\\ \psi_\omega\end{bmatrix} = J_\omega \cdot \begin{bmatrix}\varphi_\omega\\ \psi_\omega\end{bmatrix} + \begin{bmatrix}u(t)\\ 0\end{bmatrix}.$$

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Power balance. $\mathcal{E} = \frac{a}{2}|u|^2 + \frac{1}{2}\|\varphi_\omega\|^2_{L^2(\mathrm{d}\nu)} + \frac{1}{2}\|\psi_\omega\|^2_{L^2(\mathrm{d}\nu)}$ satisfies

$$\mathcal{P}(t) := \Re[(z \star u)\overline{u}](t) = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}(t).$$

$\Rightarrow$ Skew-adjoint extended evolution operator $\mathcal{A}$.

(Cassier, Joly, and Kachanovska 2017) (Gralak and Tip 2010) (Staffans 1994)

Next: dissipative realization.

Graphical motivation

Let us look at the Laplace transform of some physical models.

Unbounded cut Γ

Conjugated poles s_k

Bounded cut Γ

Bromwich contour

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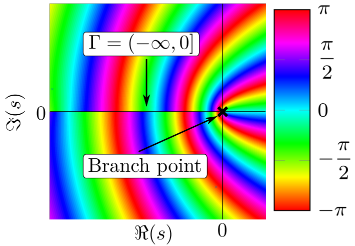

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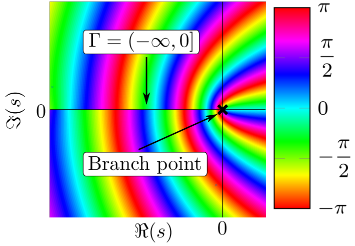

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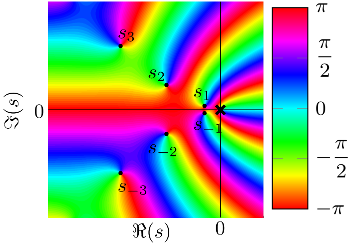

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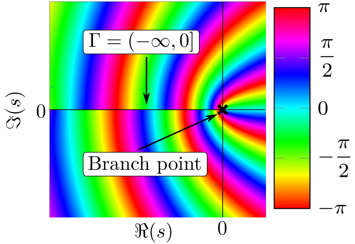

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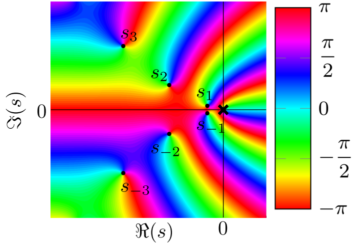

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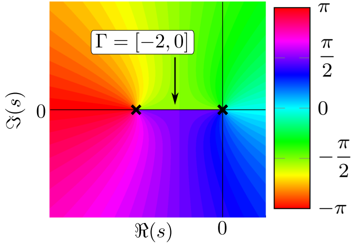

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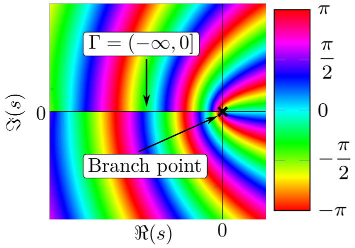

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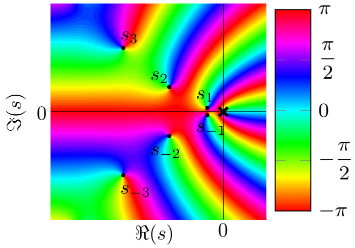

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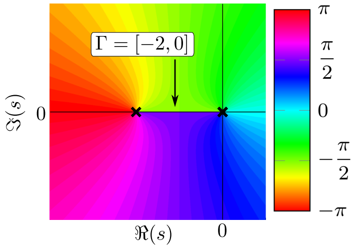

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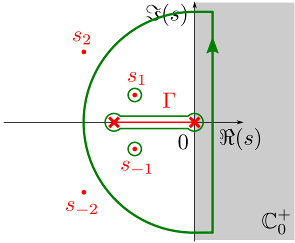


"Oscillatory-Diffusive" kernels: summary

Representation. Let $\hat{h}$ be a positive-real function.

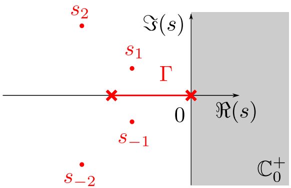
If: (i) $\hat{h}$ extends meromorphically to $\mathbb{C}\backslash\Gamma$ with $\Gamma \subset (-\infty, 0]$,
(ii) $\sup_{|s|=R} |\hat{h}(s)| \to 0$ as $R \to \infty$,
(iii) $\hat{h}$ has no poles in Int(Γ),

then

$$\hat{h}\,(s) = \sum_{k\in\mathbb{Z}} \frac{\mu_k}{s - s_k} + \int_\Gamma \frac{1}{s-\xi}\,\mathrm{d}\mu(\xi), \tag{1}$$

with measure given by jump across cut

$$\mathrm{d}\mu(\xi) = \frac{\hat{h}(|\xi|e^{-i\pi}) - \hat{h}(|\xi|e^{+i\pi})}{2i\pi}\,\mathrm{d}\xi. \tag{2}$$



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– "Diffusive" terminology: (Montseny 1998) (Hélie and Matignon 2006)
– Link with other function classes. If $\mu_k = 0$, $\mu \geq 0$, and $h(t) \in \mathbb{R}$:
- $h(t)$ is a completely monotone function (Bernstein's theorem) (Gripenberg, Londen, and Staffans 1990) (Mainardi 1997)
- $\hat{h}(s)$ is a Stieltjes function (Berg 2008)
– Link with spectral theory: (2) with $\hat{h}(s) = (A - s)^{-1}$ appears in Stone's formula (Cheverry and Raymond 2021, § 10.2).

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ODE realization:

$$h \star u(t) = \sum_{k\in\mathbb{Z}} \mu_k \varphi(t, s_k) + \int_\Gamma \varphi(t,\xi)\,\mathrm{d}\mu(\xi),\ \partial_t\varphi(t,s) = s\varphi(t,s) + u(t).$$

⚠ $\hat{h}$ multivalued $\Leftrightarrow \Gamma \neq \varnothing \Leftrightarrow$ continuum of ODEs.

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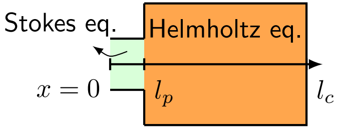
$\Rightarrow$ Dissipative extended evolution operator $\mathcal{A}$.

Next: realization of a physical model.

Dissipative realization of physical impedance models

Example: 1D modeling of a Helmholtz resonator

$$\hat{z}(s) \simeq \frac{1}{\sigma_p} \frac{\hat{p}(0)}{\hat{u}(0)}.$$



Dissipative realization of physical impedance models

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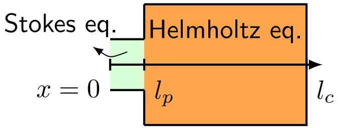
$$\hat{z}(s) \simeq \frac{1}{\sigma_p}\hat{z}_{\text{perf}}(s) + \frac{1}{\sigma_c}\coth\left(\mathrm{j}k_c\left(s\right)\, l_c\right).$$

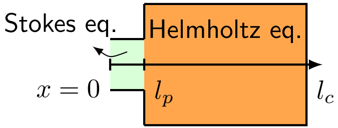


Dissipative realization of physical impedance models

Example: 1D modeling of a Helmholtz resonator

$$\hat{z}(s) \underset{|s|\to\infty}{\simeq} a_0 + a_{1/2}\sqrt{s} + a_1\, s + \frac{1}{\sigma_c}\coth\left(b_0 + b_{1/2}\sqrt{s} + b_1\, s\right),$$

with $a_{1/2}, b_{1/2} \propto \sqrt{\nu}$ (diffusion).



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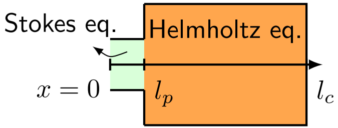

Representation is obtained by rewriting

$$\begin{cases} \hat{z}(s) = \tilde{a}_0 + a_1 s + \hat{h}_1(s) + e^{-\tau s}\,\hat{h}_2(s), \\ z \star u(t) = \tilde{a}_0 u(t) + a_1 \partial_t u + h_1 \star u(t) + h_2 \star u(t-\tau), \end{cases}$$

with delay $\tau = 2\dfrac{l_c - l_p}{c_0}$.

Realization follows from the realizations of two **irrational** kernels:

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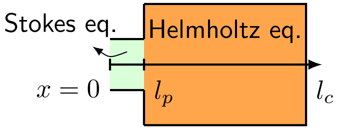

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Realization follows from the realizations of two **irrational** kernels:

"Oscillatory-diffusive" $\hat{h}(s)$.

Realization with ODE:

$$h_i \star u(t) = \int_\Gamma \varphi_s(t)\,\mathrm{d}\mu_i(s)$$

$$\partial_t \varphi_s(t) = s\,\varphi_s(t) + u(t).$$

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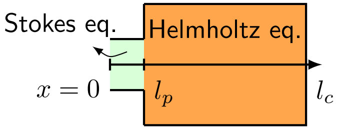

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Realization follows from the realizations of two **irrational** kernels:

Time-delay kernel $e^{-s\tau}$

Realization with transport PDE on $(-\tau, 0)$:

$$h_2 \star u(t-\tau) = \int_\Gamma \psi_s(t, -\tau)\,\mathrm{d}\mu_2(s)$$

$$\partial_t \psi_s(t,\theta) = \partial_\theta \psi_s(t,\theta),\ \psi_s(t,0) = \varphi_s(t).$$

"Oscillatory-diffusive" $\hat{h}(s)$.

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Dissipative realization: semigroup approach to stability

Extended formulation: $\partial_t X = \mathcal{A} X$ with $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$.

Asymptotic stability theorem (Arendt and Batty 1988) (Lyubich and Vũ 1988)

Assume $\mathcal{A}$ generates a C_0-semigroup of contractions $\mathcal{T}(t) \in \mathcal{L}(H)$.

If: (i) $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \varnothing$,

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then $\mathcal{T}$ is asymptotically stable:

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Example. $\hat{z}(s) = 1/\sqrt{s}$.

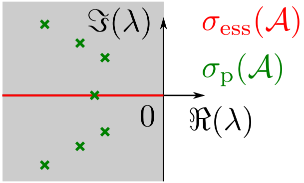
$$\mathcal{H} = \nabla H^1(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega; V_0)$$

$$\mathcal{D}(\mathcal{A}) \supset H_{\text{div}}(\Omega) \times H^1(\Omega) \times L^2(\partial\Omega; V_1),$$

with $V_s = L^2\left((0,\infty), (1+\xi)^s \, \mathrm{d}\mu(\xi)\right)$.

– No exponential stability.

– Embedding $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ not compact.



Contents

Numerical benefit of scattering formalism

Impedance formulation

$$y = \mathcal{Z}(u)$$

Absorbed energy:

$$\mathcal{E}(t) = \int_{-\infty}^{t} \Re[\mathcal{Z}(u)\overline{u}]\,\mathrm{d}\tau$$

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Numerical benefit of scattering formalism

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$y = \mathcal{Z}(u)$	$y - u = \mathcal{B}(y+u)$
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$\mathcal{E}(t) = \int_{-\infty}^{t} \Re[\mathcal{Z}(u)\overline{u}]\,\mathrm{d}\tau$	$4\,\mathcal{E}(t) = \int_{-\infty}^{t} \left[\lvert y+u\rvert^2 - \lvert\mathcal{B}(y+u)\rvert^2\right]\,\mathrm{d}\tau$
LTI case: $\mathcal{Z}(u) = z \star u$ with $\hat{z}(s)$ positive–real $(\Re[\hat{z}] \geq 0)$	LTI case: $\mathcal{B}(u) = \beta \star u$ with $\hat{\beta}(s)$ bounded–real $(\lvert\hat{\beta}\rvert \leq 1)$

Why $\mathcal{B}$? Stability, inverse methods, characteristics-based fluxes

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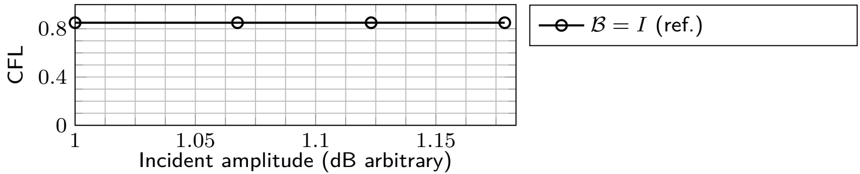
Illustration with algebraic nonlinearity (DG4, LS–ERK4(8))



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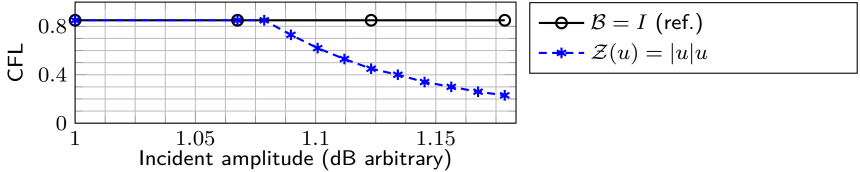

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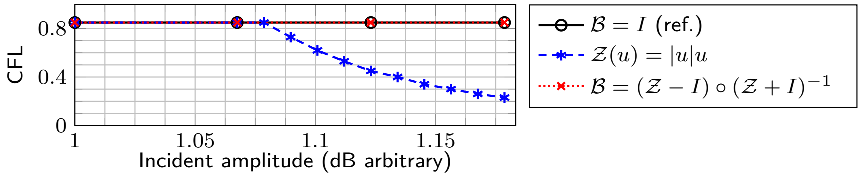
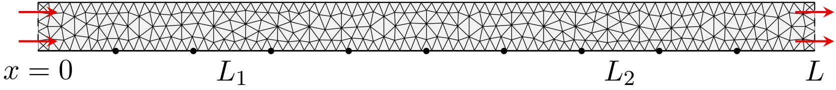
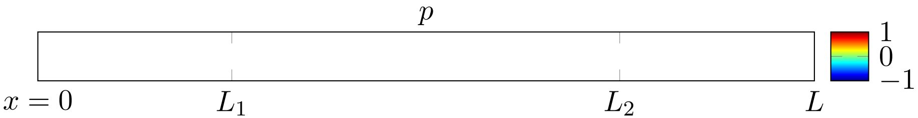
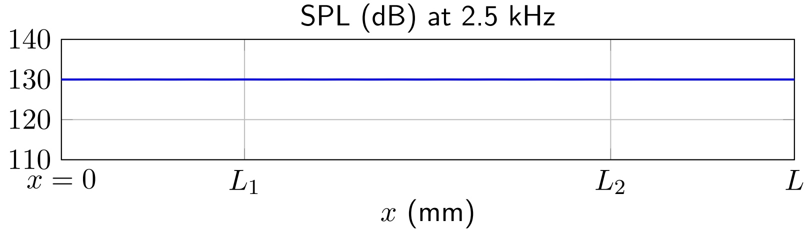


Application: linear aeroacoustics duct



DG4, 552 triangles, LS–ERK4(8), CFL=0.85





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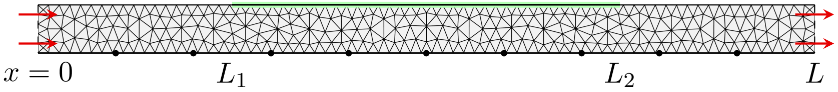

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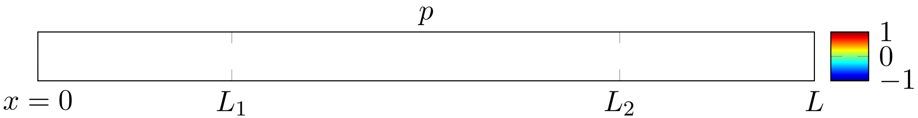
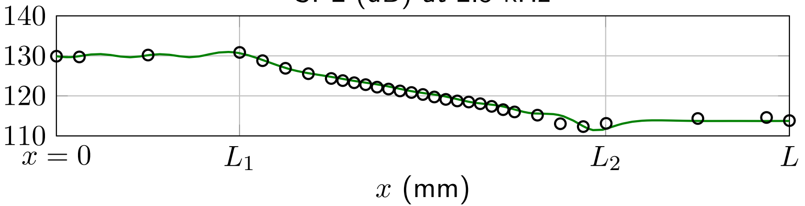


SPL (dB) at 2.5 kHz



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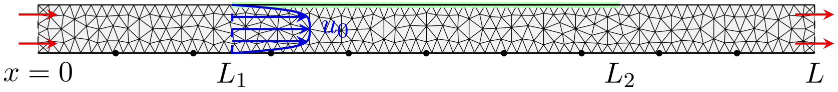

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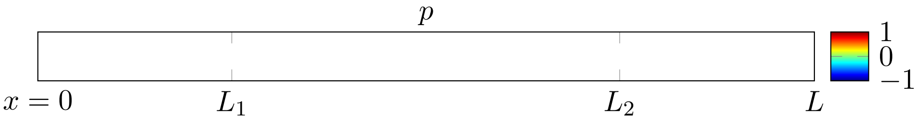
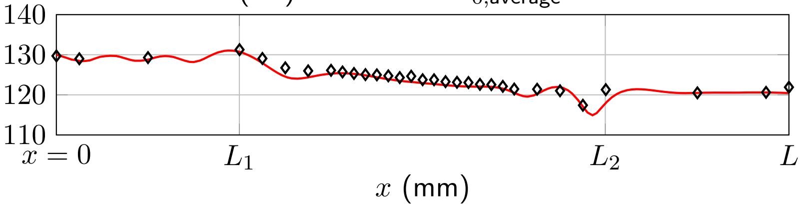


SPL (dB) at 2.5 kHz $M_{0,\text{average}} = 0.335$



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Conclusion

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Thanks for your attention.

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