Analysis and Discretization of Time-Domain Impedance Boundary Conditions in Aeroacoustics

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ONERA THE FRENCH AEROSPACE LAB

Motivation: noise reduction

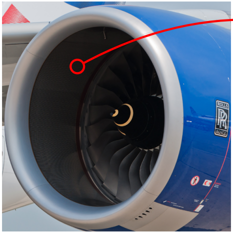

Fig. Trent 900 (A380). Inlet lined with a sound absorbing material.

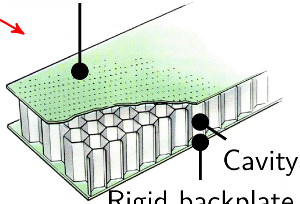

Fig. Example of liner.

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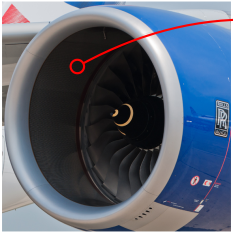

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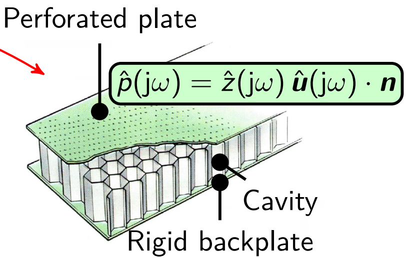

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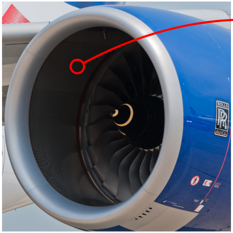

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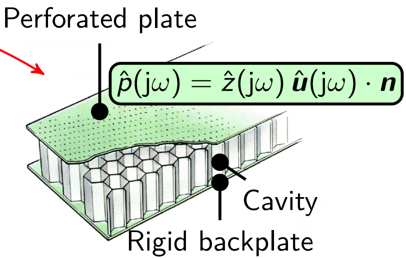

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Time-harmonic (ω) or time-domain (t) formulation?

Pros of a time-domain formulation (Tam 2012)

- Broadband sources
- Nonlinear absorption
- Nonlinear PDE (CFD)
- \+ Theoretical interest

$\Longrightarrow$ **Objective** Time-domain impedance boundary condition (IBC).

Outline

1. Introduction
   - Applicability and admissibility of IBCs
   - Existing impedance models
   - Objectives

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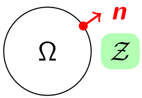
IBC: Definition, applicability, admissibility

PDE of interest

$$\partial_t \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} + \sum_{i=1}^{d} A_i \partial_{x_i} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = 0 \quad \text{on } \Omega$$

with IBC on $\partial\Omega$.



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$$p = \mathcal{Z}(\boldsymbol{u} \cdot \boldsymbol{n})$$

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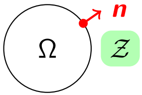

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$$p = \mathcal{Z}(\boldsymbol{u}\cdot\boldsymbol{n}) \quad \overset{\text{LTI}}{\Longrightarrow} \quad p(t) = \left[z \underset{t}{\star} \boldsymbol{u}\cdot\boldsymbol{n} \right](t)$$

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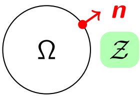

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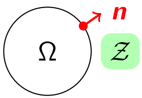

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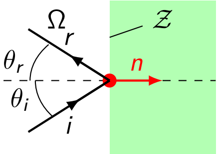
Applicability Only to locally reacting surfaces (Kinsler et al. 1962, § 6.7).



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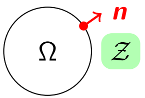

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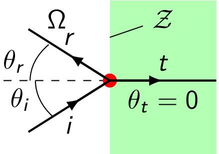

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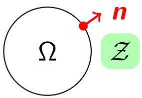

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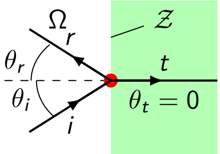

$\Rightarrow$ An **admissible** IBC dissipates energy at $\partial\Omega$.

What do impedance models look like ?

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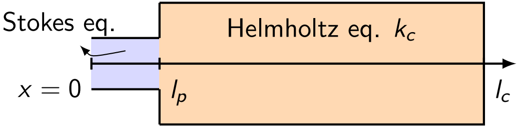
Physical models vs numerical models

Physical modeling of SDOF liner



$\xrightarrow[\text{solution}]{\text{Analytical}} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} (s, x)$

$$\hat{z}_{\text{phys}}(s) = \frac{1}{\sigma_p} \frac{\hat{p}(0)}{\hat{u}(0)}$$

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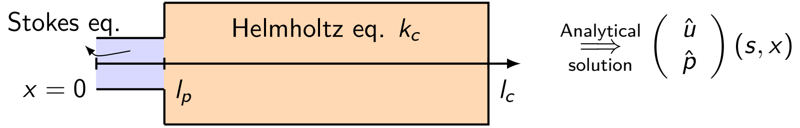

$$\hat{z}_{\text{phys}}(s) = \frac{1}{\sigma_p}\frac{\hat{p}(0)}{\hat{u}(0)} \underset{+\infty}{=} a_0 + a_{1/2}\sqrt{s} + a_1\, s + \frac{1}{\sigma_c} \coth\left(b_0 + b_{1/2}\sqrt{s} + b_1\, s\right)$$

where fractional terms are linked to viscothermal diffusion $a_{1/2}, b_{1/2} \propto \sqrt{\nu}$.

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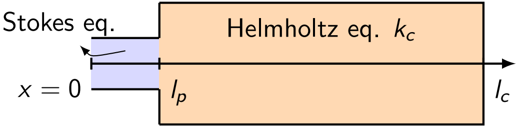

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Early works: (S. Davis 1991), (Tam et al. 1996), (Özyörük et al. 1998).

Common models **EHR** (Rienstra 2006)

$$\hat{z}_{\text{num}}(s) = a_0 + a_1 s + a_2 \coth\left(b_0 + b_1 s\right)$$

⇒ Discretization (Chevaugeon et al. 2006)

Multipole (Fung et al. 2001)

$$\hat{z}_{\text{num}}(s) = \sum_{k=1}^{N} \frac{r_k}{s - s_k}$$

⇒ "Recursive" convolution

⇒ ODE (Bin et al. 2009)

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Objectives

State of the art

- Physical $\neq$ numerical models
- Numerical models mostly linear
- Admissibility & stability explored by different communities

Objectives

Consider physical, computational, and mathematical aspects of IBCs.

(a) Time-domain structure of physical impedance models?

(b) Well-posedness and stability?

(c) Discretization?

(d) Nonlinear absorption mechanisms?

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Intro	Admissibility and examples of IBCs
Part I	Time-domain analysis of physical models
Part II	Discontinuous Galerkin discretization
Part III	Stability of wave equation

Part I: objective of model analysis

Objective Time-domain expression of linear physical models $\hat{z}_{\text{phys}}$

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Principle $\hat{z}_{\text{phys}}$ can be expressed using two simpler kernels:

Time delay	Oscillatory-diffusive (OD)
$e^{-s\tau}$	$\hat{h}(s)$

$\Rightarrow$ Enables to deduce discrete model $\hat{z}_{\text{num}}$ **from** $\hat{z}_{\text{phys}}$

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Application to a CT liner impedance model ($z_c, \sigma_c = 1$):

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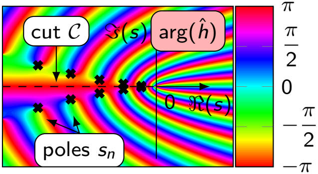
Oscillatory-diffusive representation

$$\hat{h}(s) = \overbrace{\sum_{k\in\mathbb{Z}} \frac{r_k}{s - s_k}}^{\text{oscillatory part (poles } s_k)} + \underbrace{\int_0^\infty \frac{\mu(\xi)}{s+\xi}\,\mathrm{d}\xi}_{\text{diffusive part (cut)}}$$



Outline

2. Physical impedance models in the time domain
   - Application to CT impedance model

Application to CT model: realization

Two steps to express $z_{\text{phys}} \star u(t) = u(t) + h \star u(t - \tau)$.

(1) Oscillatory-Diffusive

(2) Delay

Application to CT model: realization

Two steps to express $z_{\text{phys}} \star u(t) = u(t) + h \star u(t-\tau)$.

(1) Oscillatory-Diffusive Convolution expressed with diffusive variable φ

$$h \star u(t) = \sum_{k \in \mathbb{Z}} r_k \, \varphi(t, -s_k) + \int_0^\infty \varphi(t,\xi)\,\mu(\xi)\,\mathrm{d}\xi\,,$$

which can be computed through first-order ODEs

$$\partial_t \varphi(t,x) = -x\varphi(t,x) + u(t), \quad \varphi(t=0,x) = 0 \iff \varphi(t,x) := e^{-xt} \underset{t}{\star} u\,.$$

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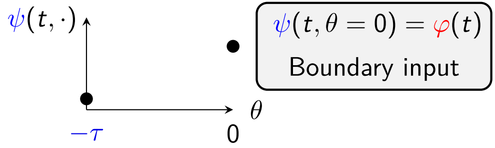

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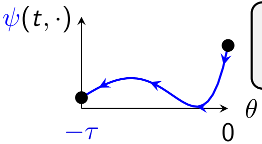

$\psi(t,\theta=0) = \varphi(t)$

Boundary input

which can be computed through 1D transport PDE on $(-\tau,0)$:

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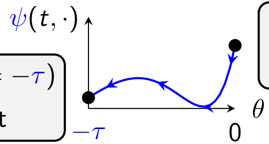
$\varphi(t-\tau) = \psi(t, \theta=-\tau)$

Boundary output



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The representation of $\hat{z}_{\text{phys}}$ suggests

$$\hat{z}_{\text{num}}(s) := 1 + e^{-\tau s}\hat{h}_{\text{num}}(s), \quad \hat{h}_{\text{num}}(s) = \sum_{k=1}^{N_s} \frac{r_k}{s - s_k} + \sum_{k=1}^{N_\xi} \frac{\mu_k}{s + \xi_k}$$

Time-local computation of $z_{\text{num}} \star u$ through

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$$J(r_k, \mu_k, \xi_k, s_k) = \sum_{k=1}^{K} |\hat{h}(\mathrm{j}\omega_k) - \hat{h}_{\text{num}}(\mathrm{j}\omega_k)|^2$$

1. Choose ξ_k, compute s_k and $r_k = \text{Res}(\hat{h}, s_k)$
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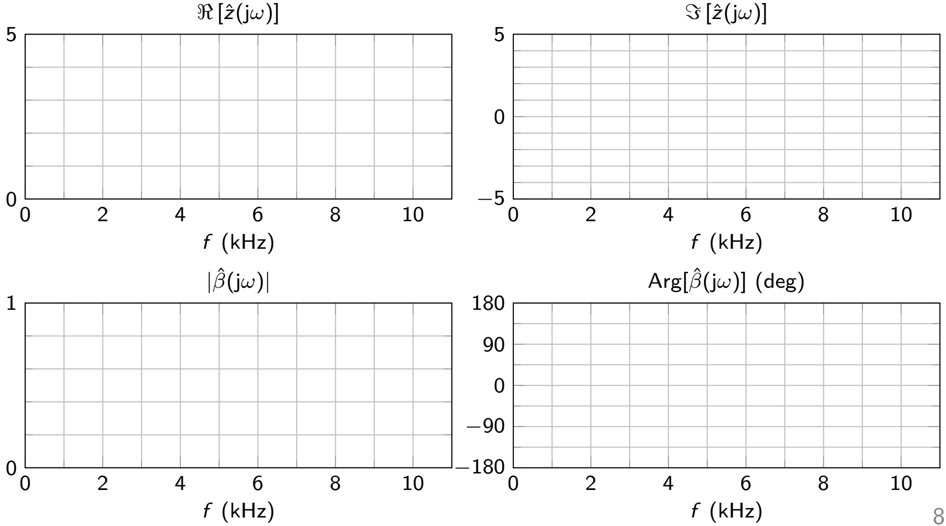
Delay Discontinuous Galerkin (DG) of order N_ψ on $(-\tau, 0)$

$$\text{PPW}(f) := \frac{N_\psi}{\tau f}$$

Application to CT model: illustration

$$\hat{z}_{\text{phys}}(j\omega) \simeq 1 + e^{-\tau j\omega}\left[\sum_{k=1}^{N_s}\frac{r_k}{j\omega - s_k} + \sum_{k=1}^{N_\xi}\frac{\mu_k}{j\omega + \xi_k}\right] =: \hat{z}_{\text{num}}(j\omega)$$



Application to CT model: illustration

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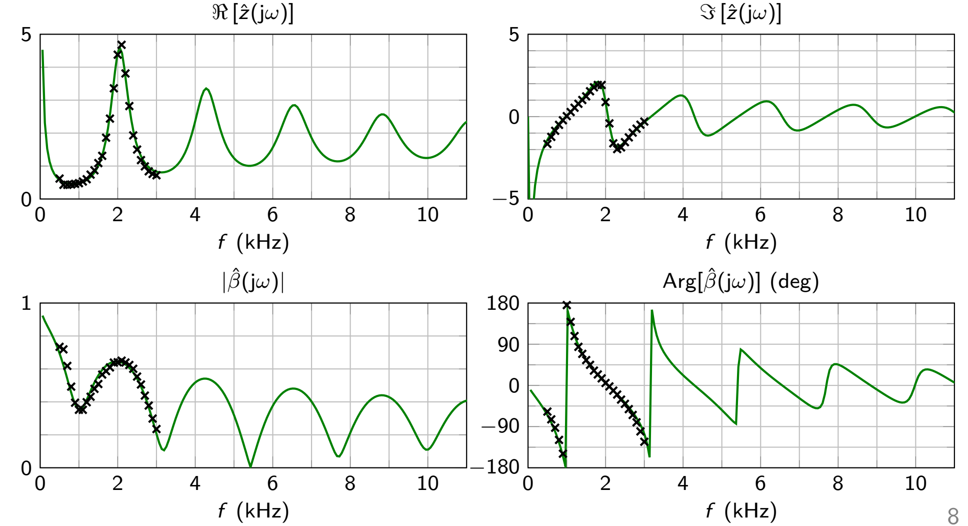

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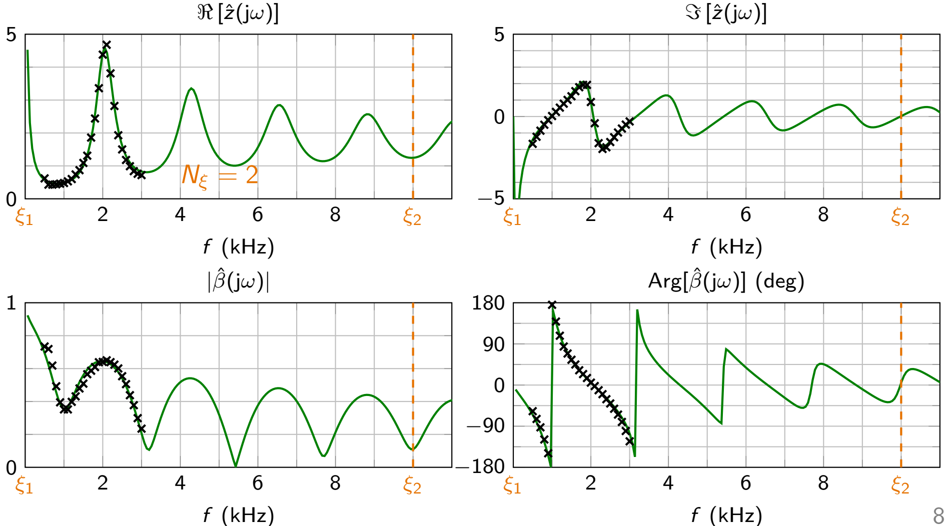

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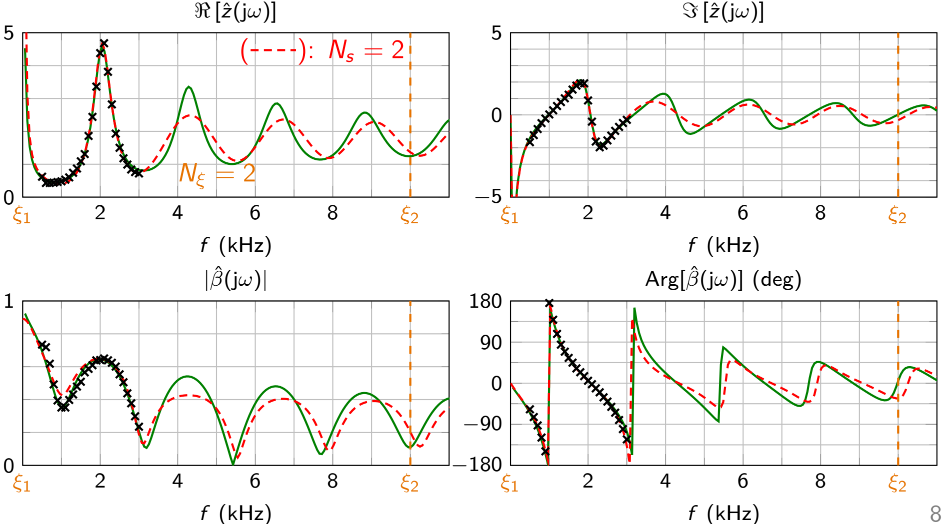

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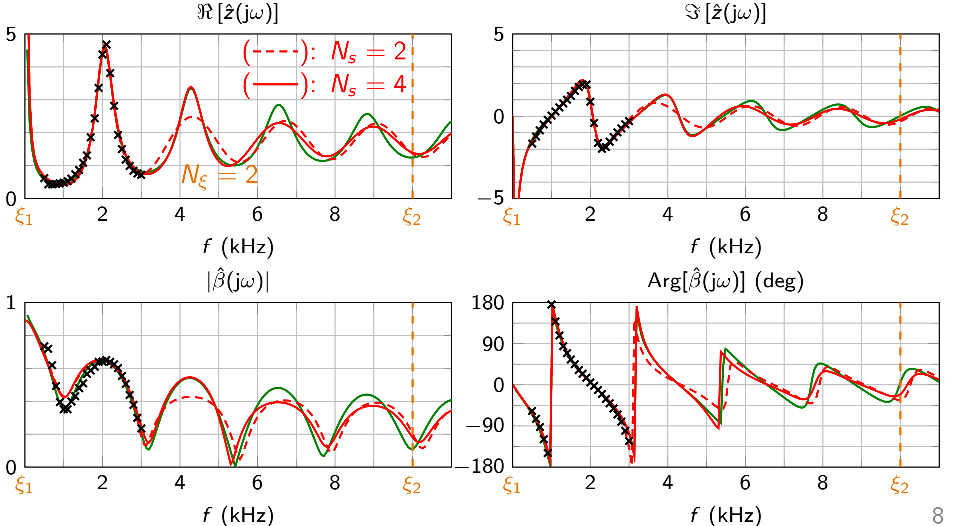

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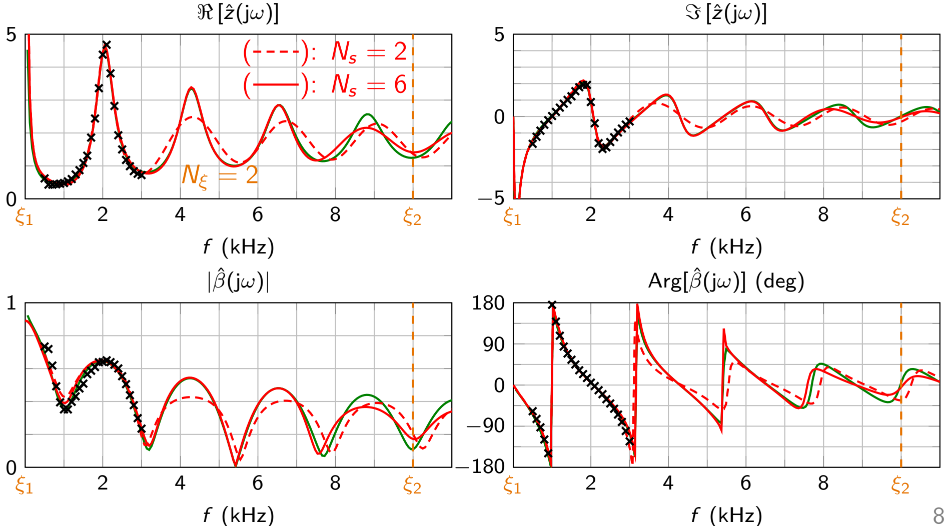

Summary of Part I: Model analysis

Questions addressed in Part I

(a) Structure of physical impedance models?

(b) Well-posedness and stability?

(c) Discretization?

(d) Nonlinear absorption mechanisms?

Further results

1. Characterization of OD kernels h
2. Discretization of OD representation (quadrature method)
3. Application to other physical models $\hat{z}_{\text{phys}}$, $\hat{y}_{\text{phys}}$, $\hat{\beta}_{\text{phys}}$

$\Rightarrow$ Part II: Discretization with Discontinuous Galerkin

Part II: Objectives

Linearized Euler equations on $(0, T) \times \Omega$, $\Omega \subset \mathbb{R}^d$

$$\begin{cases} \partial_t p + (\boldsymbol{u_0} \cdot \nabla) p + c_0 \nabla \cdot \boldsymbol{u} + \gamma p \nabla \cdot \boldsymbol{u_0} = 0 \\ \partial_t \boldsymbol{u} + (\boldsymbol{u_0} \cdot \nabla) \boldsymbol{u} + c_0 \nabla p + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u_0} + p(\boldsymbol{M_0} \cdot \nabla) \boldsymbol{u_0} = 0 \end{cases}$$

with IBC on $\Gamma_z \subset \partial\Omega$, $\boldsymbol{M_0} = \boldsymbol{u_0}/c_0$.

Objective Discretization with Discontinuous Galerkin (DG) method

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Linearized Euler equations on $(0, T) \times \Omega$, $\Omega \subset \mathbb{R}^d$

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Objective Discretization with Discontinuous Galerkin (DG) method

Continuous problem LEEs as Friedrichs system:

$$\partial_t \boldsymbol{v} + A(\nabla)\boldsymbol{v} + B\boldsymbol{v} = 0, \quad A(\boldsymbol{n}) = \begin{pmatrix} (\boldsymbol{u_0} \cdot \boldsymbol{n})\mathbb{I}_d & c_0 \boldsymbol{n} \\ c_0 \boldsymbol{n}^\mathsf{T} & \boldsymbol{u_0} \cdot \boldsymbol{n} \end{pmatrix}, \quad \boldsymbol{v} := \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix}.$$

Semi-discrete problem $\partial_t \boldsymbol{v}_h + \mathcal{A}_h \boldsymbol{v}_h = 0$, where $\mathcal{A}_h : V_h \to V_h$ is

$$(\mathcal{A}_h \boldsymbol{v}_h, \boldsymbol{w}_h)_{L^2(\Omega)} := \sum_{T \in \mathcal{T}_h} \overbrace{(\mathcal{A} \boldsymbol{v}_h, \boldsymbol{w}_h)_{L^2(T)}}^{\text{weak formulation on } T} + \underbrace{\left((A(\boldsymbol{n})\boldsymbol{v}_h)^* - A(\boldsymbol{n})\boldsymbol{v}_h, \boldsymbol{w}_h \right)_{L^2(\partial T)}}_{\text{weak coupling}}$$

Outline

3. DG discretization of IBCs
   - Energy analysis
   - Validation on nonlinear impedance tube
   - Application to duct aeroacoustics

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Weak enforcement of IBCs

Objective Definition of numerical flux $(A(\boldsymbol{n})\boldsymbol{v}_h)^*$ to weakly enforce impedance boundary condition at Γ_z?

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$\Rightarrow$ Centered flux with ghost state $\boldsymbol{v}^{\mathrm{g}}$

$$(A(\boldsymbol{n})\boldsymbol{v})^* := \frac{1}{2}A(\boldsymbol{n})\boldsymbol{v} + \frac{1}{2}A(\boldsymbol{n})\boldsymbol{v}^{\mathrm{g}}, \quad \text{with } \boldsymbol{v}^{\mathrm{g}} = \boldsymbol{v}^{\mathrm{g}}(\boldsymbol{n}, \mathcal{Z}(\boldsymbol{v}), \boldsymbol{v}).$$

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Definition of admissibility conditions

The flux $(A(\boldsymbol{n})\boldsymbol{v})^*$ is said to be *admissible* if it is both consistent and passive.

- (Consistency) Let $\boldsymbol{v}(t) \in V$ be the exact solution.

$$A(\boldsymbol{n})\boldsymbol{v}^{\mathrm{g}} = A(\boldsymbol{n})\boldsymbol{v}.$$

- (Passivity) $\forall \boldsymbol{v}_h(t) \in V_h$, $t > 0$,

$$\frac{1}{2}\int_0^t \left(A(\boldsymbol{n})\boldsymbol{v}_h^{\mathrm{g}}, \boldsymbol{v}_h\right)_{L^2(\Gamma_z)} \mathrm{d}\tau \geq 0.$$

+ desirable continuity conditions as "$\mathcal{Z}(\boldsymbol{v}) \to 0$" or "$\mathcal{Z}(\boldsymbol{v}) \to \infty$".

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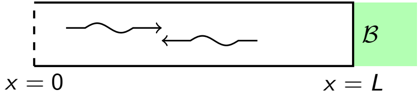
Results

- Consistent and unstable fluxes are possible
- 3 fluxes based on $\mathcal{Z}$, $\mathcal{Y}$, $\mathcal{B}$
- $\mathcal{Y}$ may be preferable to $\mathcal{Z}$
- "Ideal": scattering operator $\mathcal{B} := (\mathcal{Z} - \mathcal{I}) \circ (\mathcal{Z} + \mathcal{I})^{-1}$

Outline

3. DG discretization of IBCs
   - Energy analysis
   - Validation on nonlinear impedance tube
   - Application to duct aeroacoustics

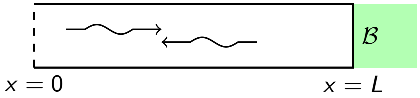
Validation: nonlinear impedance tube



Space: ArtimonDG(4)
Time: LSERK (8,4)
(Toulorge et al. 2012)

- Analytical solution even with nonlinear $\mathcal{B} \Rightarrow$ enables validation

Validation: nonlinear impedance tube



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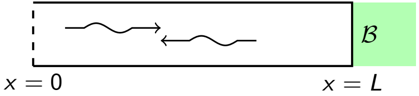
Focus on algebraic model given by $\mathcal{Z}_C(u) = a_0 u + \dfrac{C_{\text{nl}}}{c_0}|u|u$

Validation: nonlinear impedance tube



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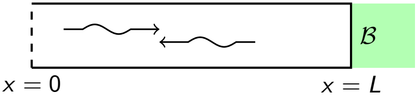
$$\mathcal{B}_C(v) = \beta_0 \frac{2v}{\Phi(v)} + \frac{C_{\text{nl}}}{c_0\,(a_0+1)^2}\frac{4|v|v}{\Phi(v)^2}, \quad \Phi(v) = 1 + \sqrt{1 + 4\frac{C_{\text{nl}}}{c_0\,(a_0+1)^2}|v|}.$$

$\Rightarrow$ $\mathcal{B}_C(v) \leq v$.

Validation: nonlinear impedance tube



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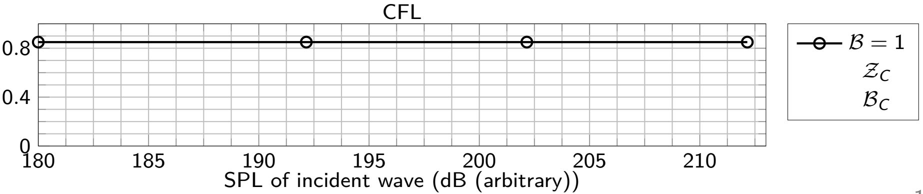
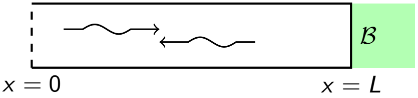
$\Rightarrow$ $\mathcal{B}_C(v) \leq v$. Differences between $\mathcal{Z}$ and $\mathcal{B}$ fluxes?



Validation: nonlinear impedance tube



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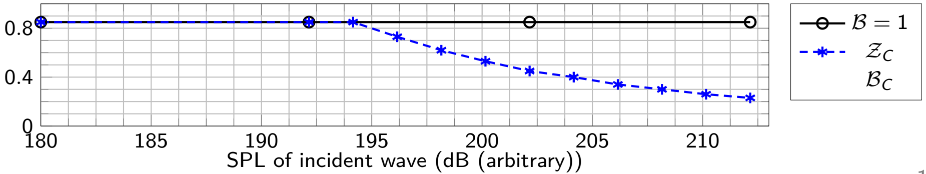

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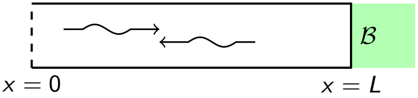

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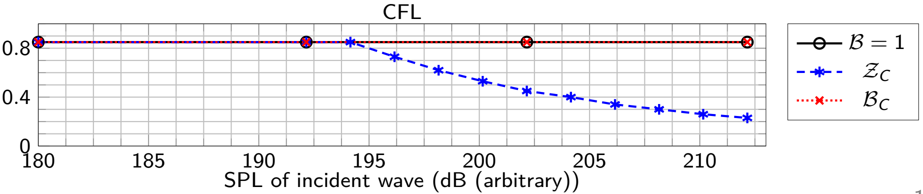
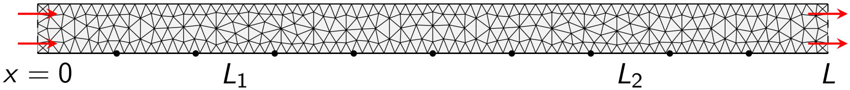


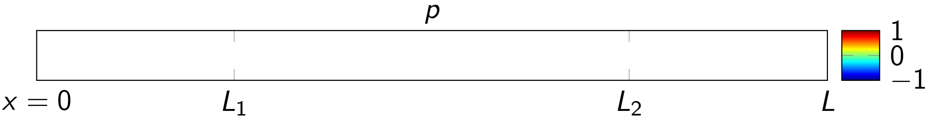
Outline

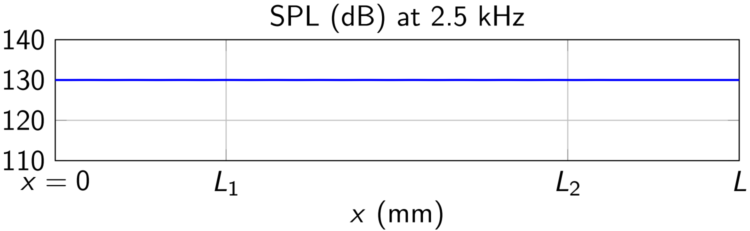
3. DG discretization of IBCs
   - Energy analysis
   - Validation on nonlinear impedance tube
   - Application to duct aeroacoustics

Aeroacoustical duct: overview



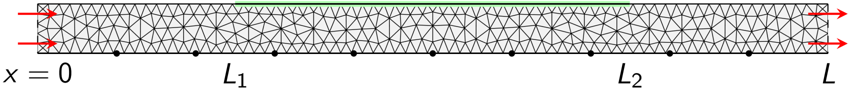
ArtimonDG(4). 552 triangles. CFL = 0.5. LSERK (8,4) (Toulorge et al. 2012)





Aeroacoustical duct: overview

$\hat{\beta}_a(s)$ (liner CT57 – NASA Langley)



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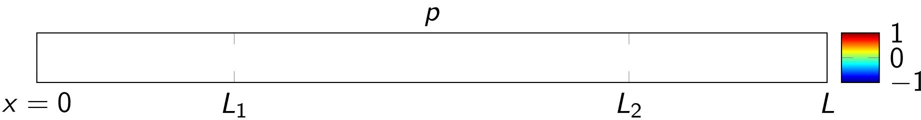

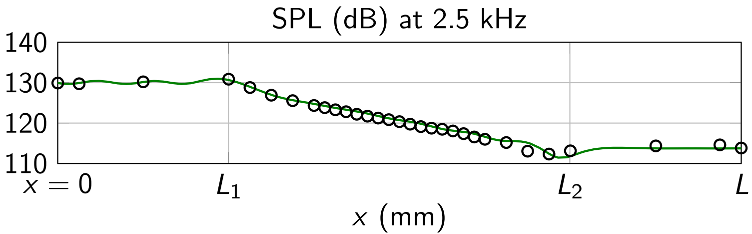

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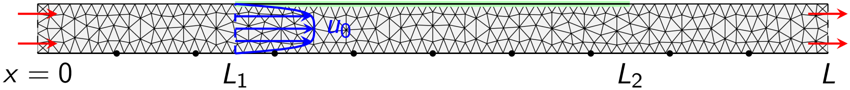

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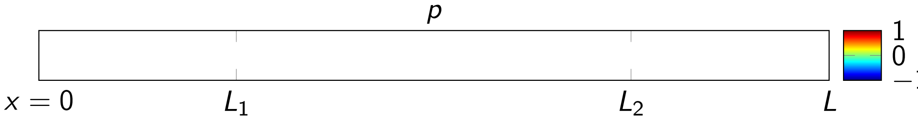
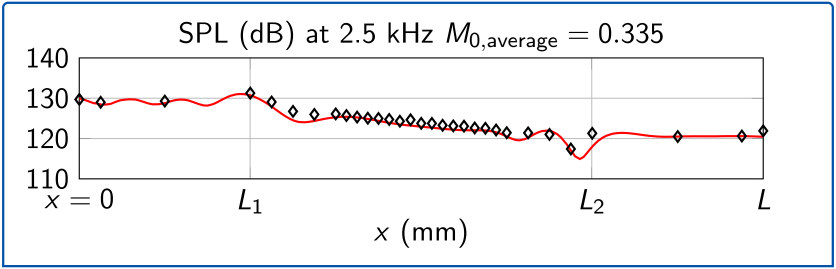




Summary of Part II: Discontinuous Galerkin discretization

Questions addressed in Part II

(a) Structure of physical impedance models?

(b) Well-posedness and stability?

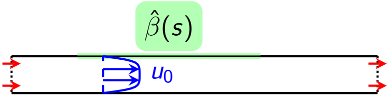
(c) Discretization?

(d) Nonlinear absorption mechanisms?

Results

1. Continuous, (Semi)-discrete energy analysis
   $\Rightarrow$ Computational advantage of β, $\mathcal{B}$ over z, $\mathcal{Z}$
2. Numerical validation on impedance tube

3. Numerical application in duct aeroacoustics



$\Rightarrow$ Part III: Stability of wave equation with IBC

Part III: Well-posedness & Stability – Summary

Model analysis is useful in both theory and practice.

Principle Representation of $\hat{z}(s)$ $\Rightarrow$ Cauchy problem on extended state

$$X = (\overbrace{p, u}^{\text{acoustic field}}, \underbrace{\varphi, \psi}_{\text{additional variables}}) \in \mathcal{H}.$$

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}\,X(t), \quad X(0) = X_0\,.$$

Objective What are we interested in?

Well-posedness and **asymptotic stability** of a strong solution X in $\mathcal{H}$.

Method of proof Spectral analysis of $\mathcal{A}$.

Asymptotic stability theorem: (Arendt et al. 1988; Lyubich et al. 1988).

Conclusions & outlook

Takeaways

(a) Structure of physical models?

- Admissibility using System theory
- Characterization of OD kernels h
- Application to physical models $\hat{z}_{\text{phys}}$, $\hat{y}_{\text{phys}}$, $\hat{\beta}_{\text{phys}}$

(b) Well-posedness and stability?

- Asymptotic stability of wave equation with positive-real IBC

(c) Discretization?

- Discretization of OD representation
- Computational advantage of β, $\mathcal{B}$ over z, $\mathcal{Z}$
- Numerical application in duct aeroacoustics

(d) Nonlinear absorption mechanisms?

- Computation of algebraic $\mathcal{B}$ and validation in nonlinear impedance tube

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Outlook

- TDIBC for DDOF liners & "exact" acoustical models

Conclusions & outlook

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Outlook

- TDIBC for DDOF liners & "exact" acoustical models
- Computation of differential $\mathcal{B}$
- Nonlinear physical modeling

Conclusions & outlook

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Outlook

- TDIBC for DDOF liners & "exact" acoustical models
- Computation of differential $\mathcal{B}$
- Nonlinear physical modeling
- Quadrature-based discretization of diffusive representations

Conclusions & outlook

Takeaways

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Outlook

- TDIBC for DDOF liners & "exact" acoustical models
- Computation of differential $\mathcal{B}$
- Nonlinear physical modeling
- Quadrature-based discretization of diffusive representations
- Stability of wave eq. with IBC
- IBCs with Euler or Navier-Stokes

Analysis and Discretization of Time-Domain Impedance Boundary Conditions in Aeroacoustics

Publications

F. Monteghetti et al. (2016). "Design of broadband time-domain impedance boundary conditions using the oscillatory-diffusive representation of acoustical models". In: *The Journal of the Acoustical Society of America* 140.3, pp. 1663–1674. DOI: 10.1121/1.4962277

F. Monteghetti et al. (2018b). "Energy analysis and discretization of nonlinear impedance boundary conditions for the time-domain linearized Euler equations". In: *Journal of Computational Physics* 375, pp. 393–426. DOI: 10.1016/j.jcp.2018.08.037

F. Monteghetti et al. (2018c). "Time-local discretization of fractional and related diffusive operators using Gaussian quadrature with applications". In: *Applied Numerical Mathematics*. DOI: 10.1016/j.apnum.2018.12.003

F. Monteghetti et al. (2018a). "Asymptotic stability of the multidimensional wave equation coupled with classes of positive real impedance boundary conditions". (In revision.)

Thanks for your attention. Any questions?

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