Analysis and Discretization of Time-Domain Impedance Boundary Conditions in Aeroacoustics

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DGA
ONERA THE FRENCH AEROSPACE LAB
isae Institut Supérieur de l'Aéronautique et de l'Espace SUPAERO

Motivation: noise reduction

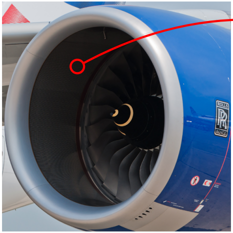

Fig. Trent 900 (A380). Inlet lined with a sound absorbing material.

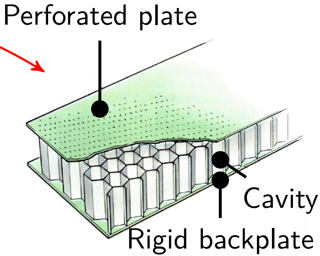

Fig. Example of liner.

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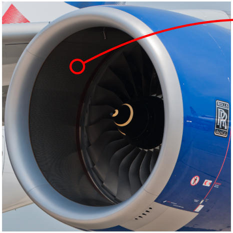

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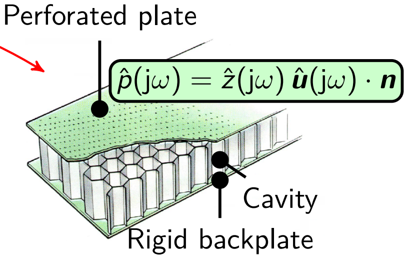

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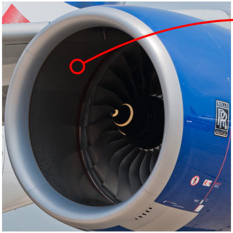

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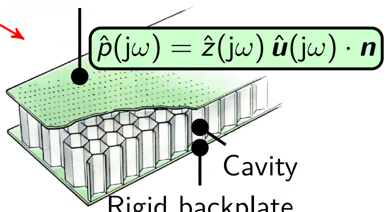

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Time-harmonic (ω) or time-domain (t) formulation?

Pros of a time-domain formulation (Tam 2012)

- Broadband sources
- Nonlinear absorption
- Nonlinear PDE (CFD)
- + Theoretical interest

$\Longrightarrow$ **Objective** Time-domain impedance boundary condition (IBC).

Outline

1. Introduction
   - Applicability and admissibility of IBCs
   - Existing impedance models
   - Objectives

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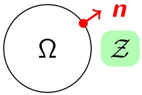
IBC: Definition and applicability

PDE of interest

$$\partial_t \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} + \sum_{i=1}^{d} A_i \partial_{x_i} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = 0 \quad \text{on } \Omega$$

with IBC on $\partial\Omega$.



Definition of an impedance boundary condition (IBC)

$$p = \mathcal{Z}(\boldsymbol{u} \cdot \boldsymbol{n})$$

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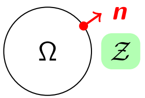

Definition of an impedance boundary condition (IBC)

$$p = \mathcal{Z}(\boldsymbol{u} \cdot \boldsymbol{n}) \quad \overset{\text{LTI}}{\Longrightarrow} \quad p(t) = \left[z \underset{t}{\star} \boldsymbol{u} \cdot \boldsymbol{n} \right] (t)$$

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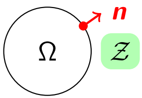

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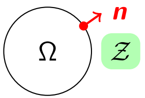

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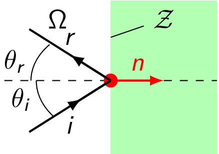
Applicability Only to locally reacting surfaces (Kinsler et al. 1962, § 6.7).



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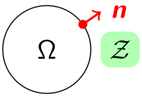

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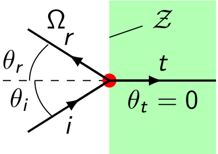

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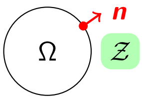

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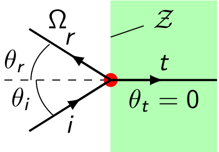

$\Rightarrow$ Properties of $\mathcal{Z}$,$\mathcal{B}$ and z, β?

IBC: Admissibility conditions

Intuition: an **admissible** IBC dissipates energy at $\partial\Omega$.

Admissibility conditions from System Theory: $u \mapsto \mathcal{Z}(u)$ is admissible if (Beltrami et al. 1966; Zemanian 1965)

- real-valued
- passive
- causal

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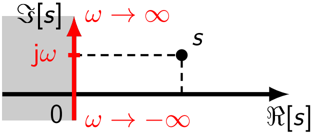
Characterization in the linear time-invariant (LTI) case

$$u \mapsto z \underset{t}{\star} u \text{ is admissible} \Leftrightarrow \hat{z}(s) \text{ is a positive-real function.}$$

$$\Leftrightarrow \hat{\beta}(s) = \frac{\hat{z}(s) - 1}{\hat{z}(s) + 1} \text{ is a bounded-real function}$$

Laplace transform:

$$\hat{z}(s) := \int_0^\infty z(t) e^{-st}\, \mathrm{d}t \quad (\Re[s] > 0)$$



$\Rightarrow$ What do impedance models look like ?

Outline

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   - Applicability and admissibility of IBCs
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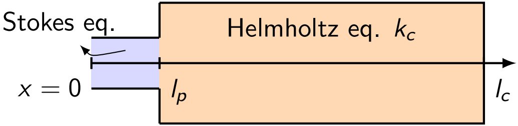
Physical models: linear acoustics

1D modeling of SDOF liner



$\overset{\text{Analytical}}{\underset{\text{solution}}{\Longrightarrow}} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix}(s, x)$

$$\hat{z}_{\text{phys}}(s) = \frac{1}{\sigma_p} \frac{\hat{p}(0)}{\hat{u}(0)}$$

,

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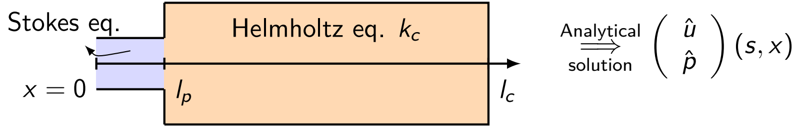

$$\hat{z}_{\text{phys}}(s) = \frac{1}{\sigma_p} \frac{\hat{p}(0)}{\hat{u}(0)} \simeq \frac{1}{\sigma_p} \hat{z}_{\text{perf}}(s) + \frac{1}{\sigma_c} \coth\left(\mathrm{j} k_c(s)\ l_c\right)$$

$$\underset{+\infty}{=} a_0 + a_{1/2}\sqrt{s} + a_1 s + \frac{1}{\sigma_c} \coth\left(b_0 + b_{1/2}\sqrt{s} + b_1 s\right),$$

where fractional terms are linked to viscothermal diffusion

$$a_{1/2}, b_{1/2} \propto \sqrt{\nu}.$$

$\Rightarrow$ Corrections

$\Rightarrow$ Expression of $\hat{\beta}_{\text{phys}}(s)$ readily follows

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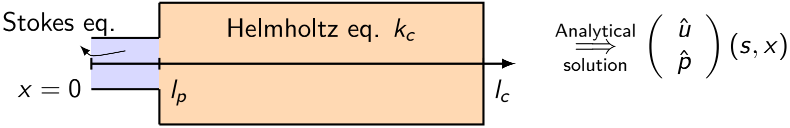

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⇒ Corrections

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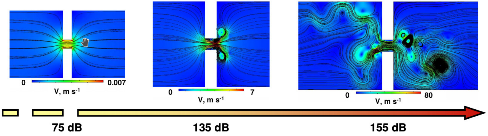
Departure from linear acoustics:

1. nonlinear absorption mechanisms
2. base flow effects (*aero*acoustics)

Physical models: nonlinear acoustics & grazing flow

Phenomenon 1 High incident amplitude ⇒ Vortex shedding



Acoustic field with increasing incident sound pressure (DNS). (Roche 2011)

Modeling

Physical models: nonlinear acoustics & grazing flow

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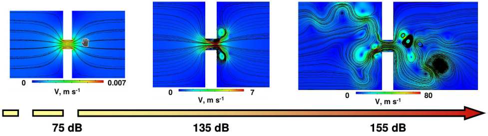

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Modeling Nonlinear operator $\mathcal{Z}$ (Cummings 1986; Meissner 1999):

$$\mathcal{Z}(\boldsymbol{u}\cdot\boldsymbol{n}) = \rho_0 C_{\mathrm{nl}} |\boldsymbol{u}\cdot\boldsymbol{n}| \boldsymbol{u}\cdot\boldsymbol{n}, \quad C_{\mathrm{nl}} \geq 0.$$

- Evidence: theoretical (Rienstra et al. 2018) & numerical (Zhang et al. 2016)
- Expression of $\mathcal{B} = (\mathcal{Z} - \mathcal{I}) \circ (\mathcal{Z} + \mathcal{I})^{-1}$

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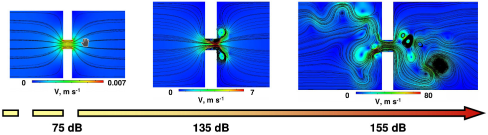

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Phenomenon 2 Grazing flow $u_0 \neq 0 \Rightarrow$ Impedance $\hat{z}_{\text{exp}}$ varies

Modeling Empirical & linear impedance correction $\hat{z}_{\text{corr}}(s, u_{0*})$ (Cummings 1986)

Working assumption IBC remains locally reacting and

$$\hat{z}_{\text{phys}}(s) = \hat{z}_{\text{phys}}(s, \boldsymbol{u}_0)$$

Numerical models

Components of a numerical "time-domain IBC" (TDIBC):

1. Discrete model $\mathcal{Z}_{\text{num}}$
2. Algorithm to evaluate $\mathcal{Z}_{\text{num}}(u)$
3. (Semi)-discrete formulation (i.e. coupling with PDE)

Challenges of numerical simulations with IBC given by $\mathcal{Z}_{\text{num}}$:

- Admissibility of $\mathcal{Z}_{\text{num}}$
- Agreement btw $\mathcal{Z}_{\text{num}}$ & $\mathcal{Z}_{\text{phys}}$
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Early works: (Davis 1991), (Tam et al. 1996), (Özyörük et al. 1998).

Common models

EHR (Rienstra 2006)

$$\hat{z}_{\text{num}}(s) = a_0 + a_1 s + a_2 \coth(b_0 + b_1 s)$$

$\Rightarrow$ Discretization (Chevaugeon et al. 2006)

Multipole (Fung et al. 2001)

$$\hat{z}_{\text{num}}(s) = \sum_{k=1}^{N} \frac{r_k}{s - s_k}$$

$\Rightarrow$ "Recursive" convolution

$\Rightarrow$ ODE (Bin et al. 2009)

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Objectives

State of the art

- Physical $\neq$ numerical models
- Numerical models mostly linear
- Admissibility & stability explored by different communities

Objectives

Consider physical, computational, and mathematical aspects of IBCs.

(a) Time-domain structure of physical impedance models?

(b) Well-posedness and stability?

(c) Discretization?

(d) Nonlinear absorption mechanisms?

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Intro	Admissibility and examples of IBCs
Part I	Time-domain analysis of physical models
Part II	Discontinuous Galerkin discretization
Part III	Stability of wave equation

Part I: objectives and contributions

Objective Time-domain expression of linear physical models $\hat{z}_{\text{phys}}$

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Objective Time-domain expression of linear physical models $\hat{z}_{\text{phys}}$

Principle $\hat{z}_{\text{phys}}$ can be expressed using two simpler kernels:

Time delay	Oscillatory-diffusive (OD)
$e^{-s\tau}$	$\hat{h}(s)$

$\Rightarrow$ Enables to deduce discrete model $\hat{z}_{\text{num}}$ **from** $\hat{z}_{\text{phys}}$

Part I: objectives and contributions

Objective Time-domain expression of linear physical models $\hat{z}_{\text{phys}}$

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Contributions of Chapter 2:

1. Characterization of OD kernels h
2. Discretization of OD kernels h (quadrature method)
3. **Application to physical models** $\hat{z}_{\textbf{phys}}$, $\hat{y}_{\text{phys}}$, $\hat{\beta}_{\text{phys}}$

Outline

2. Physical impedance models in the time domain
   - Application to CT impedance model

Application to CT model: representation

Application to a CT liner impedance model ($z_c, \sigma_c = 1$):

$$\hat{z}_{\mathrm{phys}}(s) = \coth\big(\overbrace{b_0 + b_{1/2}\sqrt{s} + b_1 s}^{:=\mathrm{j}k_c(s)}\big) \quad (\Re(s) > 0),$$

with $b_1 > 0$, $b_0, b_{1/2} \geq 0$.

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$$\hat{z}_{\text{phys}}(s) = 1 + e^{-\tau s}\hat{h}(s), \ \tau := 2b_1,$$

with $h \in \mathcal{C}((0,\infty))$ and $e^{-\kappa t} h \in L^1(0,\infty)$ for any $\kappa > 0$.

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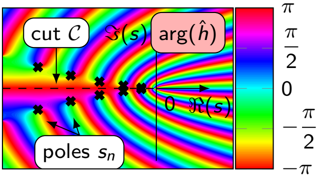
Oscillatory-diffusive representation

$$\hat{h}(s) = \overbrace{\sum_{k\in\mathbb{Z}} \frac{r_k}{s - s_k}}^{\text{oscillatory part (poles } s_k)} + \underbrace{\int_0^\infty \frac{\mu(\xi)}{s+\xi}\,\mathrm{d}\xi}_{\text{diffusive part (cut)}}$$



- 3 components: Delay / Oscillatory / Diffusive

Application to CT model: realization

Two steps to express $z_{\mathrm{phys}} \star u(t) = u(t) + h \star u(t - \tau)$.

(1) Oscillatory-Diffusive

(2) Delay

Application to CT model: realization

Two steps to express $z_{\text{phys}} \star u(t) = u(t) + h \star u(t-\tau)$.

(1) Oscillatory-Diffusive Convolution expressed with diffusive variable φ

$$h \star u(t) = \sum_{k\in\mathbb{Z}} r_k\, \varphi(t, -s_k) + \int_0^\infty \varphi(t,\xi)\,\mu(\xi)\,\mathrm{d}\xi\,,$$

which can be computed through first-order ODEs

$$\partial_t \varphi(t,x) = -x\varphi(t,x) + u(t), \quad \varphi(t=0,x) = 0 \iff \varphi(t,x) := e^{-xt} \underset{t}{\star} u\,.$$

(2) Delay

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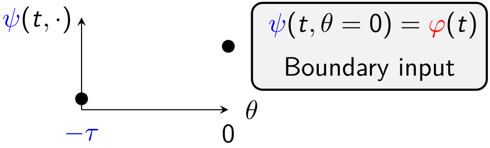
$$\partial_t \varphi(t,x) = -x\varphi(t,x) + u(t), \quad \varphi(t=0,x) = 0 \iff \varphi(t,x) := e^{-xt} \underset{t}{\star} u.$$

(2) Delay Delay expressed with hyperbolic variable ψ (x omitted)



Application to CT model: realization

Two steps to express $z_{\text{phys}} \star u(t) = u(t) + h \star u(t-\tau)$.

(1) Oscillatory-Diffusive Convolution expressed with diffusive variable φ

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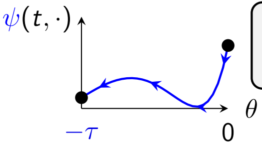

$\psi(t, \theta = 0) = \varphi(t)$

Boundary input

which can be computed through 1D transport PDE on $(-\tau, 0)$:

$$\partial_t \psi(t,\theta) = \partial_\theta \psi(t,\theta) \qquad (\theta \in (-\tau, 0))$$

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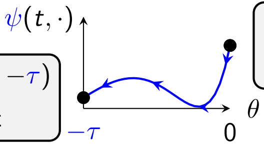
$\varphi(t-\tau) = \psi(t,\theta=-\tau)$

Boundary output



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The representation of $\hat{z}_{\text{phys}}$ suggests

$$\hat{z}_{\text{num}}(s) := 1 + e^{-\tau s}\hat{h}_{\text{num}}(s), \quad \hat{h}_{\text{num}}(s) = \sum_{k=1}^{N_s} \frac{r_k}{s - s_k} + \sum_{k=1}^{N_\xi} \frac{\mu_k}{s + \xi_k}$$

Time-local computation of $z_{\text{num}} \star u$ through

$$\text{PDE} \circ \text{ODE}, \quad (N_\psi + 1) \times (N_s + N_\xi) \text{ variables}$$

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Oscillatory-Diffusive Cost function

$$J(r_k, \mu_k, \xi_k, s_k) = \sum_{k=1}^{K} |\hat{h}(\mathrm{j}\omega_k) - \hat{h}_{\text{num}}(\mathrm{j}\omega_k)|^2$$

1. Choose ξ_k, compute s_k and $r_k = \text{Res}(\hat{h}, s_k)$
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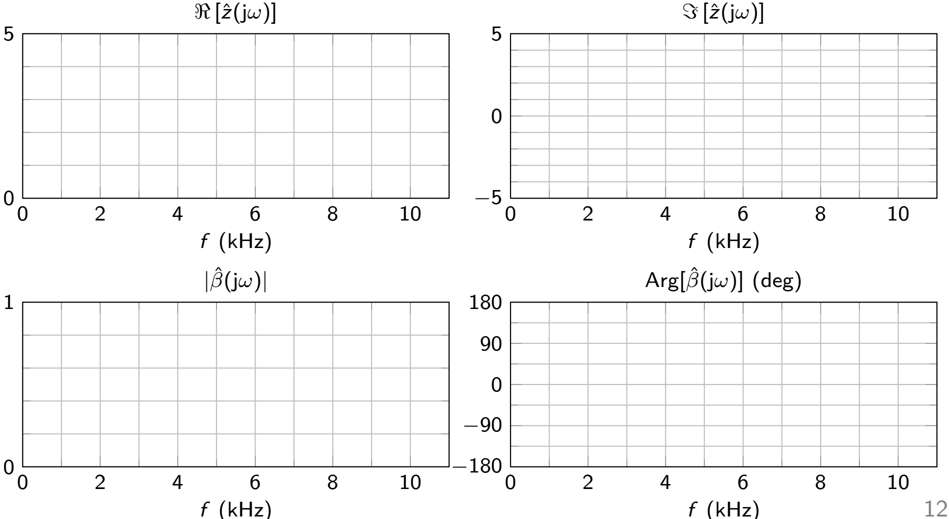
Delay Discontinuous Galerkin (DG) of order N_ψ on $(-\tau, 0)$

$$\text{PPW}(f) := \frac{N_\psi}{\tau f}$$

Application to CT model: illustration

$$\hat{z}_{\text{phys}}(\mathrm{j}\omega) \simeq 1 + e^{-\tau \mathrm{j}\omega} \left[\sum_{k=1}^{N_s} \frac{r_k}{\mathrm{j}\omega - s_k} + \sum_{k=1}^{N_\xi} \frac{\mu_k}{\mathrm{j}\omega + \xi_k} \right] =: \hat{z}_{\text{num}}(\mathrm{j}\omega)$$



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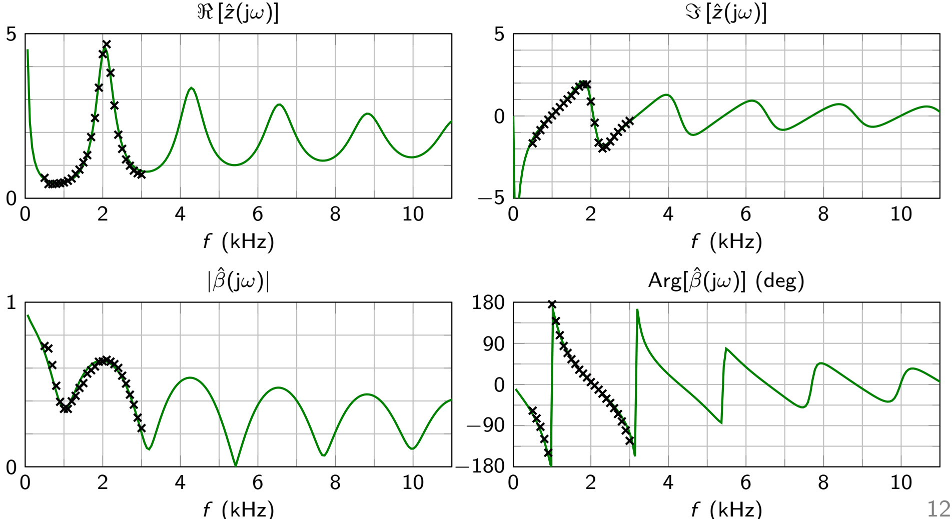
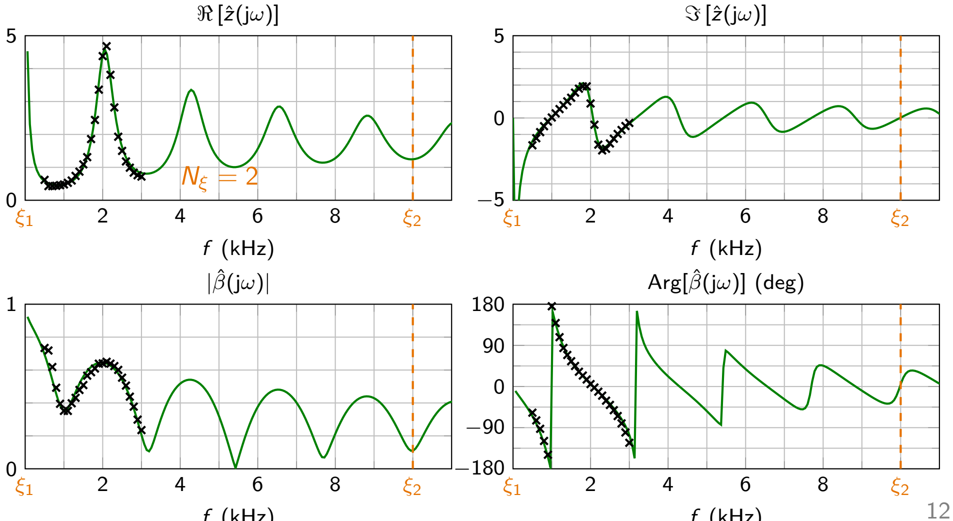


Application to CT model: illustration

$$\hat{z}_{\text{phys}}(j\omega) \simeq 1 + e^{-\tau j\omega}\left[\sum_{k=1}^{N_s} \frac{r_k}{j\omega - s_k} + \sum_{k=1}^{N_\xi} \frac{\mu_k}{j\omega + \xi_k}\right] =: \hat{z}_{\text{num}}(j\omega)$$



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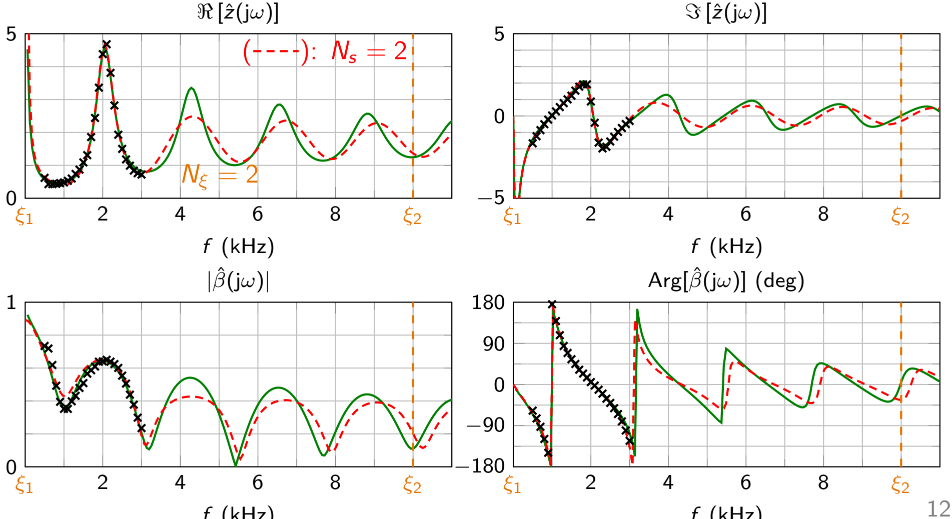

Application to CT model: illustration

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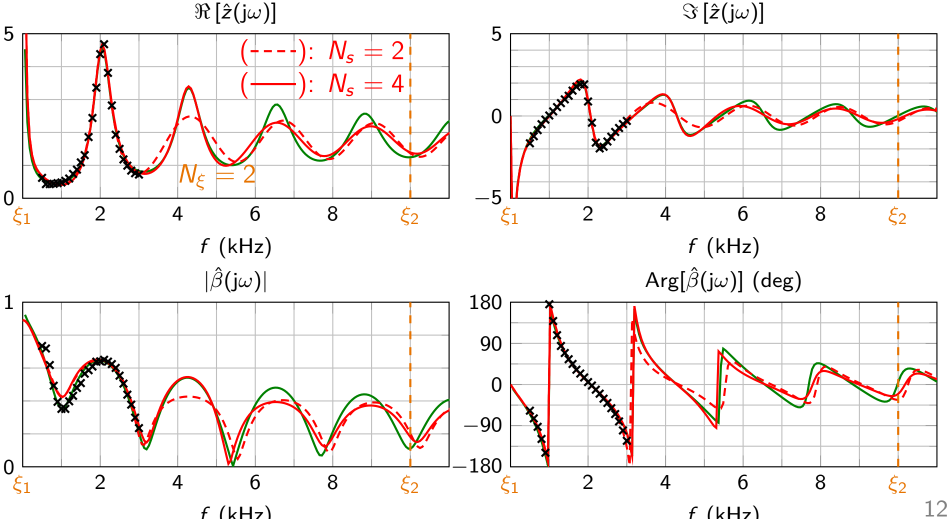

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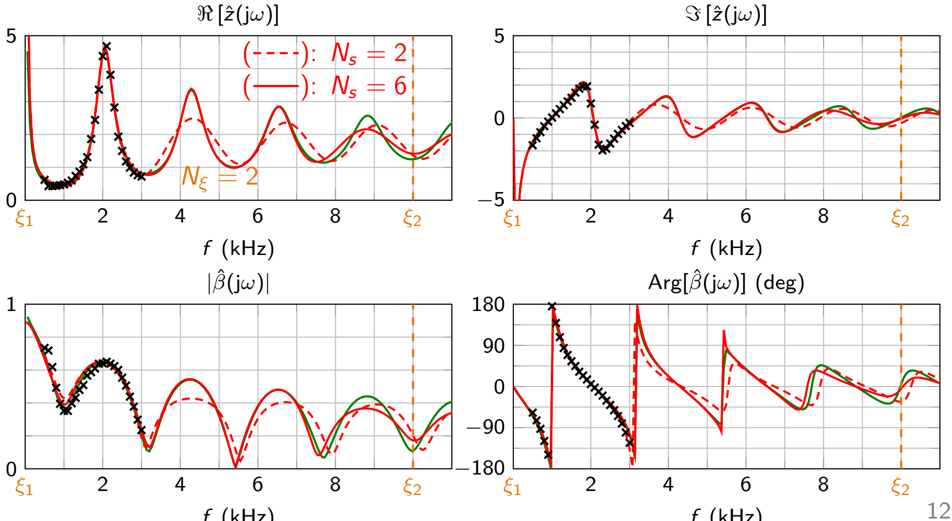

Summary of Part I: Model analysis

Questions addressed in Part I

(a) Structure of physical impedance models?

(b) Well-posedness and stability?

(c) Discretization?

(d) Nonlinear absorption mechanisms?

Contributions (Chapter 2)

1. Characterization of OD kernels h
2. Discretization of OD representation (quadrature method)
3. Application to physical models $\hat{z}_{\text{phys}}$, $\hat{y}_{\text{phys}}$, $\hat{\beta}_{\text{phys}}$

$\Rightarrow$ Part II: Discretization with Discontinuous Galerkin

Part II: Objectives and contributions

Linearized Euler equations on $(0, T) \times \Omega$, $\Omega \subset \mathbb{R}^d$

$$\begin{cases} \partial_t p + (\boldsymbol{u_0} \cdot \nabla) p + c_0 \nabla \cdot \boldsymbol{u} + \gamma p \nabla \cdot \boldsymbol{u_0} = 0 \\ \partial_t \boldsymbol{u} + (\boldsymbol{u_0} \cdot \nabla) \boldsymbol{u} + c_0 \nabla p + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u_0} + p (\boldsymbol{M_0} \cdot \nabla) \boldsymbol{u_0} = 0 \end{cases}$$

with IBC on $\Gamma_z \subset \partial\Omega$, $\boldsymbol{M_0} = \boldsymbol{u_0} / c_0$.

Objective Discretization with Discontinuous Galerkin (DG) method

Contributions of Chapters 5 and 6:

1. Continuous, (Semi)-discrete energy analysis
   $\Rightarrow$ Computational advantage of β, $\mathcal{B}$ over z, $\mathcal{Z}$

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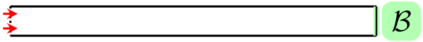
2. Numerical validation on impedance tube



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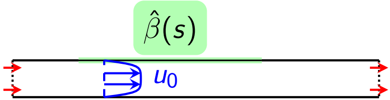

Outline

3. DG discretization of IBCs
   - Energy analysis
   - Validation on nonlinear impedance tube
   - Application to duct aeroacoustics

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Continuous formulation

LEEs written as Friedrichs system: Let $\boldsymbol{v} := \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix}$

$$\partial_t \boldsymbol{v} + A(\nabla)\boldsymbol{v} + B\boldsymbol{v} = 0, \quad A(\boldsymbol{n}) = \begin{pmatrix} (\boldsymbol{u_0} \cdot \boldsymbol{n})\mathbb{I}_d & c_0 \boldsymbol{n} \\ c_0 \boldsymbol{n}^\mathsf{T} & \boldsymbol{u_0} \cdot \boldsymbol{n} \end{pmatrix}$$

Continuous energy balance:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)} = -\frac{1}{2}(C(\boldsymbol{u_0})\boldsymbol{v}, \boldsymbol{v})_{L^2(\Omega)} - \frac{1}{2}(A(\boldsymbol{n})\boldsymbol{v}, \boldsymbol{v})_{L^2(\partial\Omega)}$$

with boundary term

$$(A(\boldsymbol{n})\boldsymbol{v}, \boldsymbol{v})_{L^2(\partial\Omega)} = \int_{\partial\Omega} (\boldsymbol{u_0} \cdot \boldsymbol{n}) \left[p^2 + |\boldsymbol{u}|^2\right] + 2c_0 \int_{\partial\Omega} p(\boldsymbol{u} \cdot \boldsymbol{n})$$

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Assumption $\boldsymbol{u_0} \cdot \boldsymbol{n} = 0$ at the impedance boundary Γ_z.

$\Rightarrow$ Due to this assumption, an admissible IBC yields

$$\int_0^t (A(\boldsymbol{n})\boldsymbol{v}, \boldsymbol{v})_{L^2(\Gamma_z)}\, \mathrm{d}\tau \geq 0 \quad (t > 0).$$

Uniqueness in $e^{-\kappa t} C((0,\infty); H^1(\Omega)^{d+1})$: well-posedness? (Chapter 3)

Discontinous Galerkin formulation

Continuous problem $\partial_t \boldsymbol{v} + \mathcal{A}\boldsymbol{v} = 0$ with $\boldsymbol{v} := \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix}$.

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$$\mathbb{P}_n^k(\mathcal{T}_h) := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_h,\ v_{|T} \in \mathbb{P}_n^k(T)\}.$$

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Semi-discrete problem Find $\boldsymbol{v}_h \in \mathcal{C}^1([0,\infty), V_h)$ such that

$$\partial_t \boldsymbol{v}_h + \mathcal{A}_h \boldsymbol{v}_h = 0,$$

where $\mathcal{A}_h :\ V_h \to V_h$ is defined by $\forall \boldsymbol{w}_h \in V_h$,

$$(\mathcal{A}_h \boldsymbol{v}_h, \boldsymbol{w}_h)_{L^2(\Omega)} := \sum_{T\in\mathcal{T}_h} \overbrace{(\mathcal{A}\boldsymbol{v}_h, \boldsymbol{w}_h)_{L^2(T)}}^{\text{weak formulation on } T} + \underbrace{\Big((A(\boldsymbol{n})\boldsymbol{v}_h)^* - A(\boldsymbol{n})\boldsymbol{v}_h, \boldsymbol{w}_h\Big)_{L^2(\partial T)}}_{\text{weak coupling}}$$

Objective Definition of numerical flux $(A(\boldsymbol{n})\boldsymbol{v}_h)^*$ to weakly enforce impedance boundary condition at Γ_z?

Weak enforcement of IBCs

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Objective Definition of numerical flux $(A(\boldsymbol{n})\boldsymbol{v}_h)^*$ to weakly enforce impedance boundary condition at Γ_z?

⇒ Centered flux with ghost state $\boldsymbol{v}^{\mathrm{g}}$

$$(A(\boldsymbol{n})\boldsymbol{v})^* := \frac{1}{2}A(\boldsymbol{n})\boldsymbol{v} + \frac{1}{2}A(\boldsymbol{n})\boldsymbol{v}^{\mathrm{g}}, \quad \text{with } \boldsymbol{v}^{\mathrm{g}} = \boldsymbol{v}^{\mathrm{g}}(\boldsymbol{n}, \mathcal{Z}(\boldsymbol{v}), \boldsymbol{v}).$$

Weak enforcement of IBCs

Objective Definition of numerical flux $(A(\boldsymbol{n})\boldsymbol{v}_h)^*$ to weakly enforce impedance boundary condition at Γ_z?

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Definition of admissibility conditions

The flux $(A(\boldsymbol{n})\boldsymbol{v})^*$ is said to be *admissible* if it is both consistent and passive.

- (Consistency) Let $\boldsymbol{v}(t) \in V$ be the exact solution.
$$A(\boldsymbol{n})\boldsymbol{v}^{\mathrm{g}} = A(\boldsymbol{n})\boldsymbol{v}.$$
- (Passivity) $\forall \boldsymbol{v}_h(t) \in V_h$, $t > 0$,
$$\frac{1}{2}\int_0^t \left(A(\boldsymbol{n})\boldsymbol{v}_h^{\mathrm{g}}, \boldsymbol{v}_h\right)_{L^2(\Gamma_z)} \mathrm{d}\tau \geq 0.$$

\+ desirable continuity conditions as "$\mathcal{Z}(\boldsymbol{v}) \to 0$" or "$\mathcal{Z}(\boldsymbol{v}) \to \infty$".

Weak enforcement of IBCs

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Results

- Consistent and unstable fluxes are possible
- 3 fluxes based on $\mathcal{Z}$, $\mathcal{Y}$, $\mathcal{B}$
- $\mathcal{Y}$ may be preferable to $\mathcal{Z}$
- "Ideal": scattering operator $\mathcal{B} := (\mathcal{Z} - \mathcal{I}) \circ (\mathcal{Z} + \mathcal{I})^{-1}$

Outline

3. DG discretization of IBCs
    - Energy analysis
    - Validation on nonlinear impedance tube
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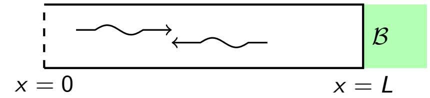
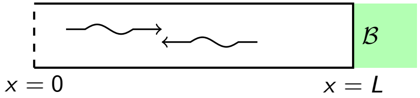
Validation: nonlinear impedance tube



Space: ArtimonDG(4)
Time: LSERK (8,4)
(Toulorge et al. 2012)

- Analytical solution even with nonlinear $\mathcal{B} \Rightarrow$ enables validation

Validation: nonlinear impedance tube



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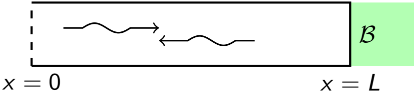
Validation: nonlinear impedance tube



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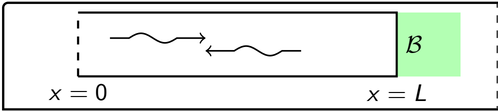
Focus on **algebraic model** given by $\mathcal{Z}_C(u) = a_0 u + \frac{C_{nl}}{c_0}|u|u$ and

$$\mathcal{B}_C(v) = \beta_0 \frac{2v}{\Phi(v)} + \frac{C_{nl}}{c_0\left(a_0+1\right)^2}\frac{4|v|v}{\Phi(v)^2}, \quad \Phi(v) = 1 + \sqrt{1 + 4\frac{C_{nl}}{c_0\left(a_0+1\right)^2}|v|}.$$

$\Rightarrow$ $\mathcal{B}_C(v) \leq v$.

Validation: nonlinear impedance tube



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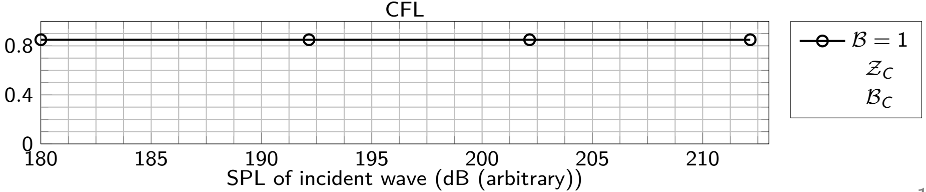
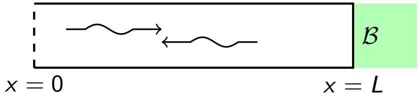
$$\mathcal{B}_C(v) = \beta_0 \frac{2v}{\Phi(v)} + \frac{C_{\text{nl}}}{c_0 \left(a_0 + 1\right)^2} \frac{4|v|v}{\Phi(v)^2}, \quad \Phi(v) = 1 + \sqrt{1 + 4 \frac{C_{\text{nl}}}{c_0 \left(a_0 + 1\right)^2} |v|}.$$

$\Rightarrow$ $\mathcal{B}_C(v) \leq v$. Differences between $\mathcal{Z}$ and $\mathcal{B}$ fluxes?



Validation: nonlinear impedance tube



Space: ArtimonDG(4)
Time: LSERK (8,4)
(Toulorge et al. 2012)

- Analytical solution even with nonlinear $\mathcal{B}$ $\Rightarrow$ enables validation

---

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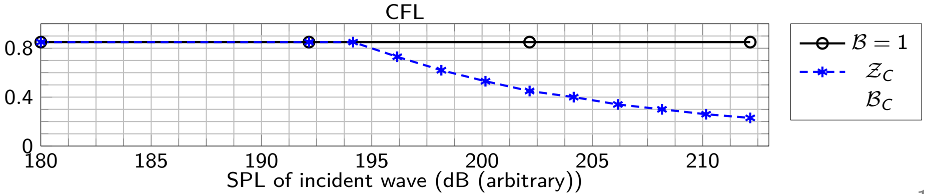

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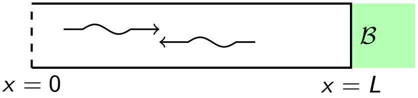

Space: ArtimonDG(4)
Time: LSERK (8,4)
(Toulorge et al. 2012)

- Analytical solution even with nonlinear $\mathcal{B}$ $\Rightarrow$ enables validation

---

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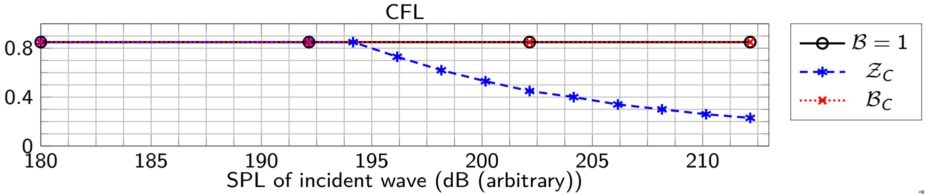
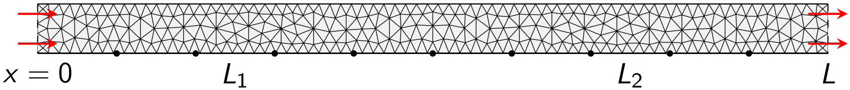


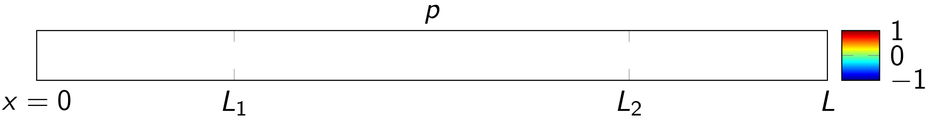
Outline

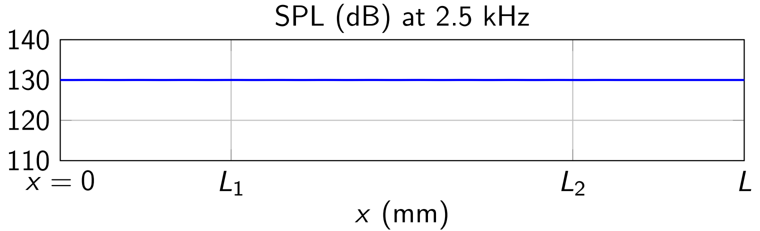
3 DG discretization of IBCs
- Energy analysis
- Validation on nonlinear impedance tube
- Application to duct aeroacoustics

Aeroacoustical duct: overview



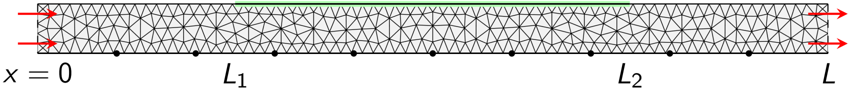
ArtimonDG(4). 552 triangles. CFL = 0.5. LSERK (8,4) (Toulorge et al. 2012)





Aeroacoustical duct: overview

$\hat{\beta}_a(s)$ (liner CT57 – NASA Langley)



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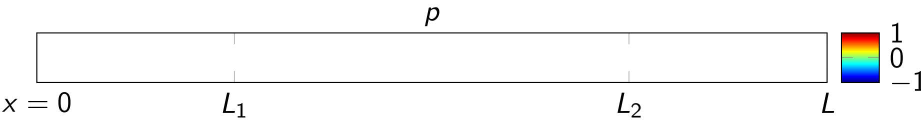

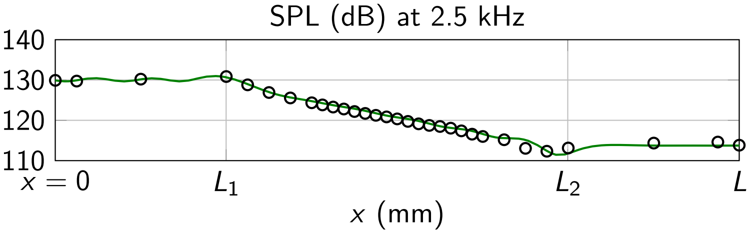

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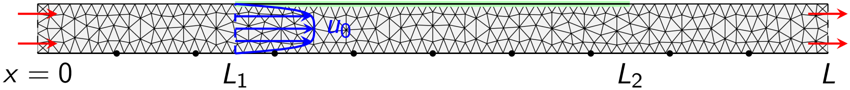

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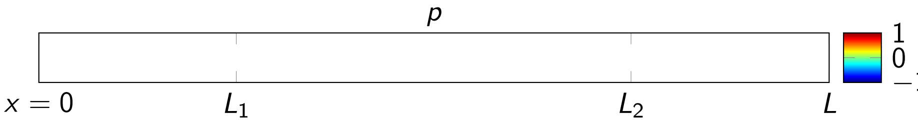
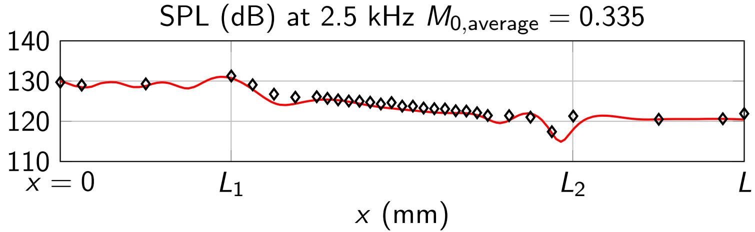




Summary of Part II: Discontinuous Galerkin discretization

Questions addressed in Part II

(a) Structure of physical impedance models?

(b) Well-posedness and stability?

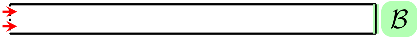
(c) Discretization?

(d) Nonlinear absorption mechanisms?

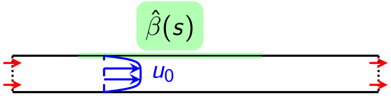
Contributions (Chapters 5&6)

1. Continuous, (Semi)-discrete energy analysis
   $\Rightarrow$ Computational advantage of β, $\mathcal{B}$ over z, $\mathcal{Z}$

2. Numerical validation on impedance tube



3. Numerical application in duct aeroacoustics



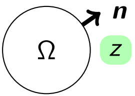
$\Rightarrow$ Part III: Stability of wave equation with IBC

Part III: objectives and contributions

Cauchy problem Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^\infty$ bounded open set.

$$\partial_t \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} + \begin{pmatrix} \nabla p \\ \operatorname{div} \boldsymbol{u} \end{pmatrix} = \mathbf{0}$$

with $p = z \underset{t}{\star} \boldsymbol{u} \cdot \boldsymbol{n}$ on $\partial\Omega$.



Contribution Asymptotic stability: $\forall X_0,\ \|X(t)\|_H \xrightarrow{t\to\infty} 0$

Strategy (Intuition)

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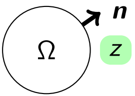

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$$\hat{z}(s) = \left(z_0 + z_\tau e^{-\tau s}\right) + z_1 s + \hat{Z}(s) + \hat{z}_{\text{diff},1}(s) + s\,\hat{z}_{\text{diff},2}(s) \quad (\Re(s) > 0)$$

where $\hat{Z}$ is strictly proper rational and $\hat{z}_{\text{diff},i} \in L^1_{\text{loc}}$ completely monotone.

Limitation Each term is positive-real: $\tau > 0,\ z_\tau \in \mathbb{R},\ z_0 \geq |z_\tau|,\ z_1 > 0$.

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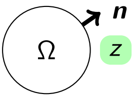

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Strategy (Intuition)

1. Find dynamical system in state-space Φ to compute $z \star \boldsymbol{u} \cdot \boldsymbol{n}$
2. Formulate an extended Cauchy problem
   $\dot{X} = \mathcal{A}X$, with extended state $X = (\boldsymbol{u}, p, \varphi) \in L^2(\Omega)^{n+1} \times L^2(\partial\Omega; \Phi)$.
3. Study energy balance: $\dot{\mathcal{E}} = \dot{\mathcal{E}}_{\text{ac}} + \dot{\mathcal{E}}_\Phi \leq 0$, use Lümer-Phillips.
4. Inspect $\sigma(\mathcal{A})$, if needed for stability.

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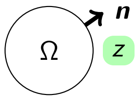

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Outline

4. Stability of wave equation with IBCs
   - Diffusive impedance
   - Delay impedance

Outline

4 Stability of wave equation with IBCs
- Diffusive impedance
- Delay impedance

Standard diffusive impedance: setup

Kernel Let $\mu \geq 0$ Radon measure s.t. $\int (1+\xi)^{-1} \mathrm{d}\mu(\xi) < \infty$.

$$\hat{z}(s) := \hat{z}_{\text{diff}}(s) = \int_0^\infty \frac{1}{s+\xi} \mathrm{d}\mu(\xi).$$

Setup

Standard diffusive impedance: setup

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$$V_s := \left\{ \varphi : (0,\infty) \to \mathbb{C} \text{ measurable} \;\middle|\; \int_0^\infty |\varphi(\xi)|^2 (1+\xi)^s \, \mathrm{d}\mu(\xi) < \infty \right\}.$$

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$$X := \begin{pmatrix} \boldsymbol{u} \\ p \\ \varphi \end{pmatrix}, \quad \mathcal{A}X := \begin{pmatrix} -\nabla p \\ -\operatorname{div} \boldsymbol{u} \\ -\xi\varphi + \boldsymbol{u} \cdot \boldsymbol{n} \end{pmatrix}$$

$$\mathcal{D}(\mathcal{A}) := \left\{ (\boldsymbol{u}, p, \varphi) \in H \;\middle|\; \begin{array}{l} (\boldsymbol{u}, p, \varphi) \in H_{\text{div}}(\Omega) \times H^1(\Omega) \times L^2(\partial\Omega; V_1) \\ (-\xi\varphi + \boldsymbol{u} \cdot \boldsymbol{n}) \in L^2(\partial\Omega; V_0) \\ p = \int \varphi \, \mathrm{d}\mu \text{ in } H^{\frac{1}{2}}(\partial\Omega) \end{array} \right\}$$

Standard diffusive impedance: stability

3 steps to prove **Asymptotic stability** (Arendt et al. 1988; Lyubich et al. 1988)

❶ "$\mathcal{A}$ is dissipative".

❷ "$\mathcal{A}$ is injective".

❸ "$s\mathcal{I} - \mathcal{A}$ is bijective for $s \in (0, \infty) \cup \mathrm{j}\mathbb{R}^*$".

Standard diffusive impedance: stability

3 steps to prove **Asymptotic stability** (Arendt et al. 1988; Lyubich et al. 1988)

1. "$\mathcal{A}$ is dissipative". For $\mathcal{E}(t) := \|(\boldsymbol{u}, p, \varphi)\|_H^2$,

$$\dot{\mathcal{E}}(t) = (\mathcal{A}X, X)_H = -\int_0^\infty \xi \|\varphi(\cdot, \xi)\|^2_{L^2(\partial\Omega)} \,\mathrm{d}\mu(\xi) \leq 0.$$

2. "$\mathcal{A}$ is injective".

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2. "$\mathcal{A}$ is injective". Crucially relies on Hodge decomposition:

$$H_{\mathrm{div}\,0,0}(\Omega) \cap \nabla H^1(\Omega) = \{0\}, \quad \text{since} \quad (L^2(\Omega))^d = \nabla H^1(\Omega) \oplus H_{\mathrm{div}\,0,0}(\Omega).$$

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3. "$s\mathcal{I} - \mathcal{A}$ is bijective for $s \in (0, \infty) \cup \mathrm{j}\mathbb{R}^*$". Key step is result below.

Proposition

Let $\hat{z}$ be a positive-real function. Then, $\forall l \in H^{-1}(\Omega)$ and $s \in \mathbb{C}^*$ such that $\Re(s) \geq 0$, there is a unique $p \in H^1(\Omega)$ solving

$$\forall \psi \in H^1(\Omega),\ (\nabla p, \nabla \psi) + s^2 (p, \psi) + \frac{s}{\hat{z}(s)} (p, \psi)_{L^2(\partial\Omega)} = \overline{l(\psi)}, \tag{1}$$

with $\|p\|_{H^1(\Omega)} \leq C(s) \, \|l\|_{H^{-1}(\Omega)}$.

Ingredients of proof. Fredholm alternative and Rellich identity.

Outline

4. Stability of wave equation with IBCs
   - Diffusive impedance
   - Delay impedance

Delay impedance: overview

Kernel $\hat{z}(s) = z_0 + z_\tau e^{-\tau s}$ (positive-real iff $z_0 \geq |z_\tau|$)

$\Rightarrow$ Expression of $z \star u$ in $L^2(-\tau, 0)$ through transport equation.

Setup

Asymptotic stability

Delay impedance: overview

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$\Rightarrow$ Expression of $z \star u$ in $L^2(-\tau, 0)$ through transport equation.

Setup Energy space $H := \nabla H^1(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega; L^2(-\tau, 0))$

$$X := \begin{pmatrix} \boldsymbol{u} \\ p \\ \psi \end{pmatrix}, \quad \mathcal{A}X := \begin{pmatrix} -\nabla p \\ -\operatorname{div} \boldsymbol{u} \\ \partial_\theta \psi \end{pmatrix}$$

$$\mathcal{D}(\mathcal{A}) := \left\{ (\boldsymbol{u}, p, \psi) \in H \;\middle|\; \begin{array}{l} (\boldsymbol{u}, p) \in H_{\mathrm{div}}(\Omega) \times H^1(\Omega) \\ \psi \in L^2(\partial\Omega; H^1(-\tau, 0)) \\ p = z_0 \boldsymbol{u} \cdot \boldsymbol{n} + z_\tau \psi(\cdot, -\tau) \text{ in } L^2(\partial\Omega) \\ \psi(\cdot, 0) = \boldsymbol{u} \cdot \boldsymbol{n} \text{ in } L^2(\partial\Omega) \end{array} \right\}$$

Asymptotic stability

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Asymptotic stability

1. "$\mathcal{A}$ is dissipative". This holds iff $z_0 \geq |z_\tau|$.

Summary of Part III: Stability of wave equation with IBC

Questions addressed in Part II

(a) Structure of physical impedance models?

(b) Well-posedness and stability?

(c) Discretization?

(d) Nonlinear absorption mechanisms?

Contribution Asymptotic stability: $\forall X_0,\ \|X(t)\|_H \xrightarrow{t\to\infty} 0$ with

$$\hat{z}(s) = \left(z_0 + z_\tau e^{-\tau s}\right) + z_1 s + \hat{Z}(s) + \hat{z}_{\text{diff},1}(s) + s\,\hat{z}_{\text{diff},2}(s) \quad (\Re(s) > 0)$$

where $\hat{Z}$ is strictly proper rational and $\hat{z}_{\text{diff},i} \in L^1_{\text{loc}}$ completely monotone.

$\Rightarrow$ Overall conclusion

Main contributions & outlook

Contributions

(a) Structure of physical models?

- Admissibility using System theory
- Characterization of OD kernels h
- Application to physical models $\hat{z}_{\text{phys}}$, $\hat{y}_{\text{phys}}$, $\hat{\beta}_{\text{phys}}$

(b) Well-posedness and stability?

- Asymptotic stability of wave equation with positive-real IBC

(c) Discretization?

- Discretization of OD representation
- Computational advantage of β, $\mathcal{B}$ over z, $\mathcal{Z}$
- Numerical application in duct aeroacoustics

(d) Nonlinear absorption mechanisms?

- Computation of algebraic $\mathcal{B}$ and validation in nonlinear impedance tube

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Outlook

- TDIBC for DDOF liners & "exact" acoustical models

Main contributions & outlook

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- Computation of algebraic $\mathcal{B}$ and validation in nonlinear impedance tube

Outlook

- TDIBC for DDOF liners & "exact" acoustical models
- Computation of differential $\mathcal{B}$
- Nonlinear physical modeling

Main contributions & outlook

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(a) Structure of physical models?

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- Characterization of OD kernels h
- Application to physical models $\hat{z}_{\text{phys}}$, $\hat{y}_{\text{phys}}$, $\hat{\beta}_{\text{phys}}$

(b) Well-posedness and stability?

- Asymptotic stability of wave equation with positive-real IBC

(c) Discretization?

- Discretization of OD representation
- Computational advantage of β, $\mathcal{B}$ over z, $\mathcal{Z}$
- Numerical application in duct aeroacoustics

(d) Nonlinear absorption mechanisms?

- Computation of algebraic $\mathcal{B}$ and validation in nonlinear impedance tube

Outlook

- TDIBC for DDOF liners & "exact" acoustical models
- Computation of differential $\mathcal{B}$
- Nonlinear physical modeling
- Quadrature-based discretization of diffusive representations

Main contributions & outlook

Contributions

(a) Structure of physical models?

- Admissibility using System theory
- Characterization of OD kernels h
- Application to physical models $\hat{z}_{\text{phys}}$, $\hat{y}_{\text{phys}}$, $\hat{\beta}_{\text{phys}}$

(b) Well-posedness and stability?

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Outlook

- TDIBC for DDOF liners & “exact” acoustical models
- Computation of differential $\mathcal{B}$
- Nonlinear physical modeling
- Quadrature-based discretization of diffusive representations
- Stability of wave eq. with IBC
- IBCs with Euler or Navier-Stokes

Communications & Publications (1)

Publications

F. Monteghetti et al. (2016a). "Design of broadband time-domain impedance boundary conditions using the oscillatory-diffusive representation of acoustical models". In: *The Journal of the Acoustical Society of America* 140.3, pp. 1663–1674. DOI: 10.1121/1.4962277

F. Monteghetti et al. (2018b). "Energy analysis and discretization of nonlinear impedance boundary conditions for the time-domain linearized Euler equations". In: *Journal of Computational Physics* 375, pp. 393–426. DOI: 10.1016/j.jcp.2018.08.037

Publications (submitted)

F. Monteghetti et al. (2018a). "Asymptotic stability of the multidimensional wave equation coupled with classes of positive real impedance boundary conditions". (Submitted.)

F. Monteghetti et al. (2018d). "Time-local discretization of fractional and related diffusive operators using Gaussian quadrature with applications". (Submitted.)

Communications & Publications (2)

Communications

F. Monteghetti et al. (2016c). "Simulation temporelle d'un modèle d'impédance de liner en utilisant la représentation diffusive d'opérateurs". In: *13e Congrès Français d'Acoustique*. (Le Mans, France). 000130, pp. 2549–2555

F. Monteghetti et al. (2016b). "High-order time-domain simulation of acoustic impedance models using diffusive representation". In: *Poster session of the XVII Spanish-French School Jacques-Louis Lions about Numerical Simulation in Physics and Engineering*. (Gijón, Spain)

F. Monteghetti et al. (2017b). "Asymptotic stability of the linearised Euler equations with long-memory impedance boundary condition". In: *13th International Conference on Mathematical and Numerical Aspects of Wave Propagation (WAVES 2017)*. (Minneapolis, MN, USA)

F. Monteghetti et al. (2017a). "Stability of linear fractional differential equations with delays: a coupled parabolic-hyperbolic PDEs formulation". In: *20th World Congress of the International Federation of Automatic Control (IFAC)*. (Toulouse, France). DOI: 10.1016/j.ifacol.2017.08.1966

F. Monteghetti et al. (2018c). "Quadrature-based diffusive representation of the fractional derivative with applications in aeroacoustics and eigenvalue methods for stability". In: *10th Workshop Structural Dynamical Systems: Computational Aspects (SDS2018)*. (Capitolo (Monopoli), Italy)

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2 Physical impedance models in the time domain

3 DG discretization of IBCs

4 Stability of wave equation with IBCs

5 Conclusion

Thanks for your attention. Any questions?

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Additional slides