Quadrature-based diffusive representation of the fractional derivative with applications

in aeroacoustics and eigenvalue methods for stability

10th Workshop SDS, Capitolo, Italy — 13 June 2018

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Outline

Motivation: fractional delay systems in aeroacoustics

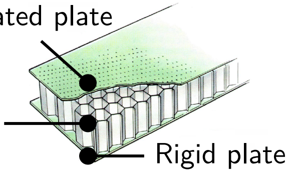
Context: Noise regulations ⇒ research into sound absorption.

Modeling of locally-reacting sound absorbing material

Passive LTI system:

$$p(t,x) = \left[z \underset{t}{\star} \boldsymbol{u} \cdot \boldsymbol{n}(\cdot, x)\right](t)$$

with kernel $z \in \mathcal{D}'_+(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$.



Key components of z: (Monteghetti et al. 2016)

$$\hat{z}(s) = 1 + \hat{h}(s) + e^{-s\tau}$$

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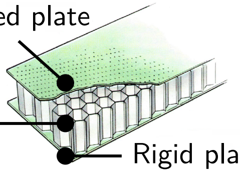

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$$\hat{z}(s) = 1 + \hat{h}(s) + e^{-s\tau}$$

$\Rightarrow$ Boundary condition of a PDE on (p, u).

$\Rightarrow$ Spatial discretization yields memory delay equation ($x \in \mathbb{R}^n$):

$$M \cdot \dot{x}(t) + K \cdot x(t) = F_1 \cdot h \star x(t) + F_2 \cdot x(t-\tau).$$

Objective: discretization of a time-local representation of $h \star x(t)$.

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Background on diffusive representation: definition

Definition (Desch et al. 1988; Staffans 1994; Mainardi 1997; Montseny 1998)

A causal kernel $h \in L^1_{\mathrm{loc}}([0,\infty))$ is *standard diffusive* if

$$h(t) = \int_0^\infty e^{-\xi t}\,\mu(\xi)\,\mathrm{d}\xi\,, \qquad \hat{h}(s) = \int_0^\infty \frac{1}{s+\xi}\,\mu(\xi)\,\mathrm{d}\xi\,,$$

with weight $\mu \in \mathcal{C}((0,\infty))$.

Computational interest. Time-local computation of convolution through

$$h \star u(t) = \int_0^\infty \varphi(t,\xi)\,\mu(\xi)\,\mathrm{d}\xi$$

with state $\varphi(t,\cdot) \in L^2(0,\infty;\mu(\xi)\mathrm{d}\xi)$ that obeys

$$\begin{cases} \partial_t \varphi(t,\xi) = -\xi\,\varphi(t,\xi) + u(t) \\ \varphi(0,\cdot) = 0. \end{cases}$$

Background on diffusive representation: examples

Let $\alpha \in (0,1)$.

Riemann-Liouville integral and Caputo derivative of order α

$I^\alpha u := Y_\alpha \star u$ with $u \in L^2$.	$\mathrm{d}^\alpha u := I^{1-\alpha} \dot{u}$ with $u \in H^1$.

Fractional integral (Montseny 1998)

$$I^\alpha u(t) = \int_0^\infty \varphi(t,\xi)\, \mu_\alpha(\xi)\, \mathrm{d}\xi, \qquad \mu_\alpha(\xi) = \frac{\sin(\alpha\pi)}{\pi \xi^\alpha}.$$

Similar for Caputo derivative (Lombard et al. 2016)

$$\mathrm{d}^\alpha u(t) = \int_0^\infty \underbrace{[-\xi\, \varphi(t,\xi) + u(t)]}_{=\partial_t \varphi(t,\xi)}\, \mu_{1-\alpha}(\xi)\, \mathrm{d}\xi, \qquad \varphi(0,\xi) = \frac{u(0)}{\xi}.$$

Also: Bessel function J_0. (Matignon 1998, § 3.3)

Discretization?

Discretization of diffusive representation: state of the art

$$h(t) = \int_0^\infty e^{-\xi t}\,\mu(\xi)\,\mathrm{d}\xi \simeq h_{\mathrm{num}}(t) := \sum_{n=1}^{N} \mu_n e^{-\xi_n t}$$

Optimization-based

Quadrature-based

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- Linear least squares (Garcia et al. 1998; Hélie et al. 2006)
- Nonlinear least squares (Lombard et al. 2016)

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Optimization-based

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Quadrature-based

- Truncation on $(10^a, 10^b)$ with Gauss-Legendre and Curtis-Clenshaw (Baranowski 2017)
- Fast time stepping: split of $Y_\alpha \star u(t)$ between "local" and "historical" parts (Haddar et al. 2010; Li 2010)
- Gauss-Laguerre on $(0, \infty)$ (Yuan et al. 2002)
- Gauss-Jacobi on $(-1, 1)$ (Diethelm 2008) (Birk et al. 2010)

Quadrature-based discretization: definition

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Strategy? (Davis et al. 1984, Chap. 3) (Atkinson 1989, § 5.6) (Shampine 2008, § 4.2)

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Strategy? (Davis et al. 1984, Chap. 3) (Atkinson 1989, § 5.6) (Shampine 2008, § 4.2) Change of variables $\xi = \Psi(v)$

$$h(t) = \int_{-1}^{1} \mu(\Psi(v)) e^{-\Psi(v)t}\Psi'(v)\,\mathrm{d}v \Rightarrow \begin{cases} \xi_n := \Psi(v_n) \\ \mu_n := w_n \Psi'(v_n)\mu(\xi_n) \end{cases}$$

with (v_n, w_n) Gauss-Legendre nodes.

Quadrature-based discretization: definition

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Assumption

$$\mu \text{ is non-oscillating} \quad \text{and} \quad \mu(\xi) \underset{0}{=} \mathcal{O}\left(\frac{1}{\xi^\alpha}\right)$$

Definition of $Q_{\beta,N}$ method

$$\Psi_\beta(v) := \left(\frac{1+v}{1-v}\right)^{\frac{1}{\beta}} \quad \text{with} \quad \beta > 0$$

Properties? Choice of β?

Quadrature-based discretization: analysis

$$h(t) = \frac{2}{\beta}\int_{-1}^{1} e^{-\Psi_\beta(v)t}(1-v)^{-1-\frac{1}{\beta}}(1+v)^{\frac{1}{\beta}-1}\mu(\Psi_\beta(v))\,\mathrm{d}v$$

Best β? $\Rightarrow$ Integrand regularity. (Atkinson 1989, Thm. 5.4) (Dunham 1930, Thm. I.VIII)

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Results for $h = Y_\alpha$, $\alpha \in (0,1)$

$$\beta \leq 1-\alpha \quad \Rightarrow \quad Y_{\alpha,\mathrm{num}}(t) \to Y_\alpha(t) \quad (t>0)$$

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$$\beta \leq \min(\alpha, 1-\alpha) \quad \Rightarrow \quad \|Y_{\alpha,\text{num}}\|_{L^1(0,T)} \to \|Y_\alpha\|_{L^1(0,T)}$$

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$\beta \leq 1-\alpha$	$\Rightarrow$	$Y_{\alpha,\text{num}}(t) \to Y_\alpha(t) \quad (t>0)$
$\beta \leq \min(\alpha, 1-\alpha)$	$\Rightarrow$	$\lVert Y_{\alpha,\text{num}} \rVert_{L^1(0,T)} \to \lVert Y_\alpha \rVert_{L^1(0,T)}$
$(+)\ \frac{1}{\beta} \in \mathbb{N}, \quad \frac{\alpha}{\beta} \in \mathbb{N}$	$\Rightarrow$	Spectral convergence

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Conclusion: two choices for β

$\alpha = {}^{n_1}/_{n_2}$ or $\alpha \simeq {}^{n_1}/_{n_2}$

$\Rightarrow \beta_1 := {}^{1}/_{n_2}$

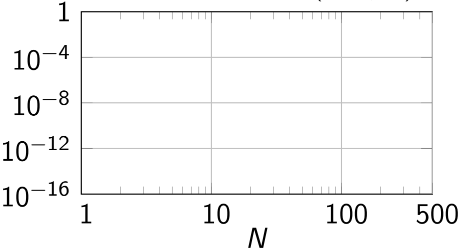
$\beta_2 := \min(\alpha, 1-\alpha)$

Let's test this numerically!

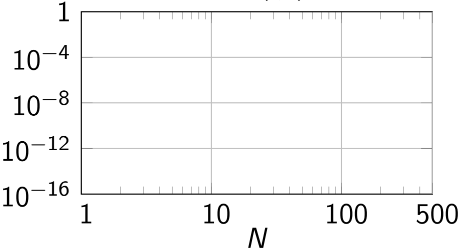
Quadrature-based discretization: illustration

$$h = Y_{\frac{1}{2}}, \quad \omega_m = 10^4, \quad T = 10^4.$$

$$\left\| 1 - \frac{\hat{h}_{\text{num}}}{\hat{h}}(i\omega) \right\|_{L^\infty(-\omega_m,\omega_m)}$$



$$\frac{\left| \|h\|_{L^1(0,T)} - \|h_{\text{num}}\|_{L^1(0,T)} \right|}{\|h\|_{L^1(0,T)}}$$



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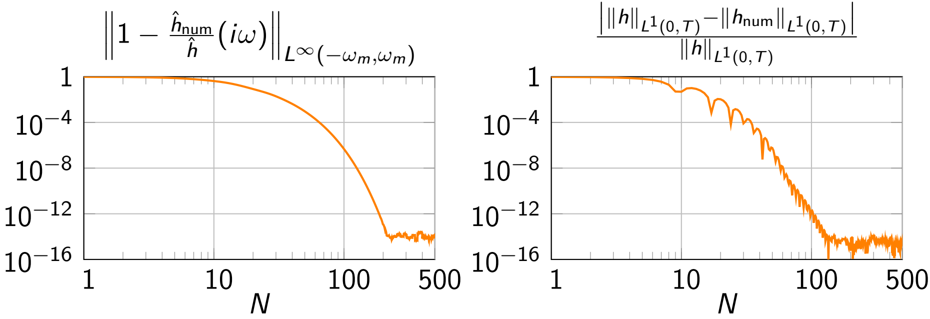

$Q_{\beta,N}$ discretization with (——) $\beta = \beta_1$

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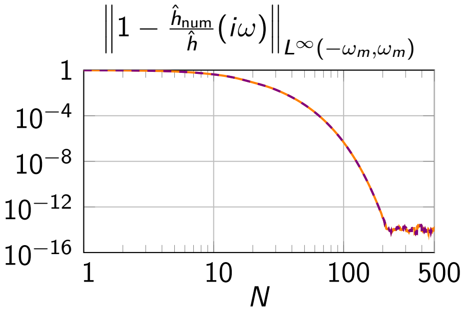

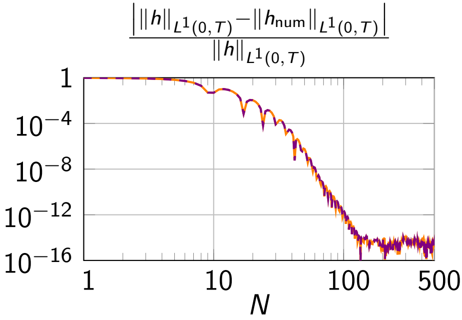

$Q_{\beta,N}$ discretization with (——) $\beta = \beta_1 \Leftrightarrow$ (- - -) $\beta = \beta_2 = \frac{1}{2}$

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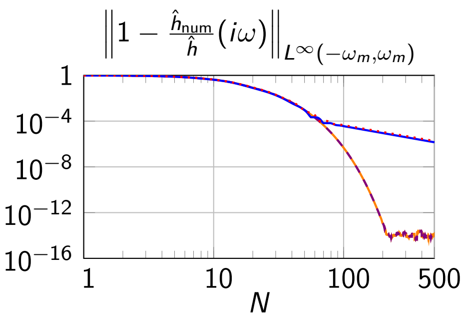

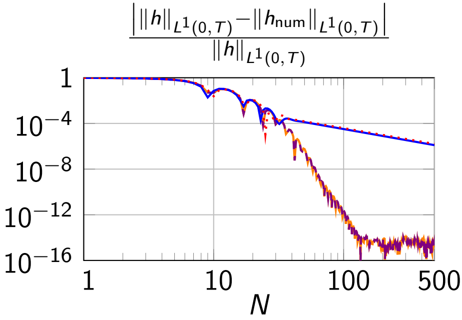

$Q_{\beta,N}$ discretization with (—) $\beta = \beta_1 \Leftrightarrow$ (- - -) $\beta = \beta_2 = \frac{1}{2}$

(—) $\beta = \beta_2 \times 0.99$
(⋯) $\beta = \beta_2 \times 1.01$ } Sensitivity to β

Quadrature-based discretization: illustration (2)

If $\alpha \neq \frac{1}{2}$, $\beta_1 \neq \beta_2$. Let

$$h = Y_{\frac{5}{8}}, \quad \omega_m = 10^4, \quad T = 10^4.$$

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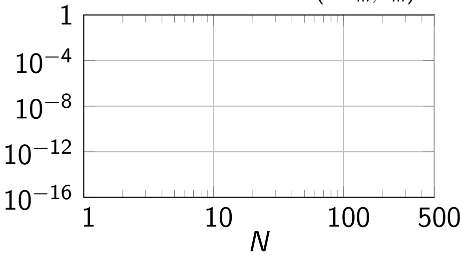

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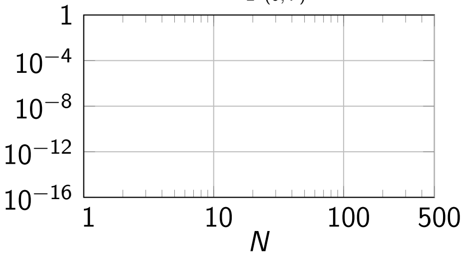

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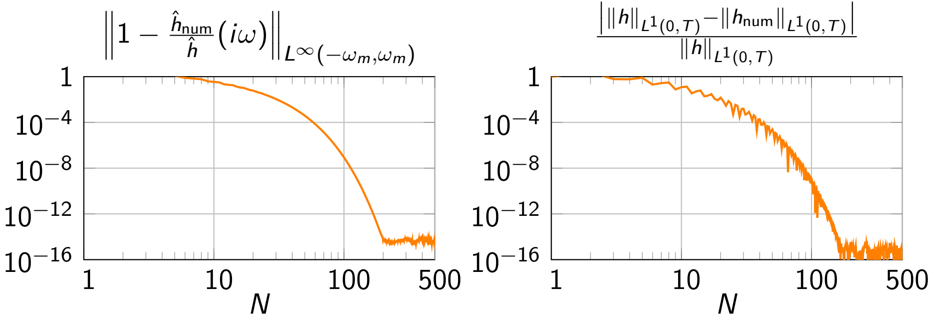

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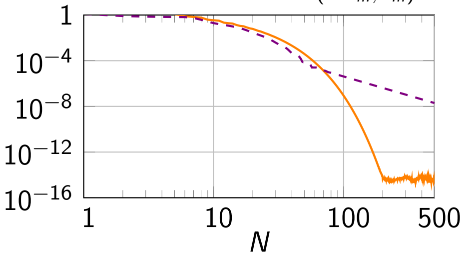

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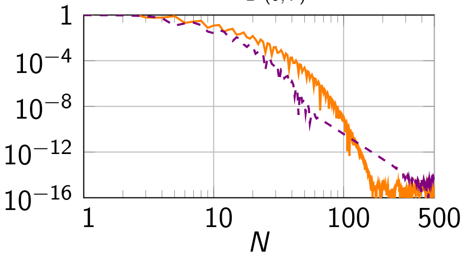

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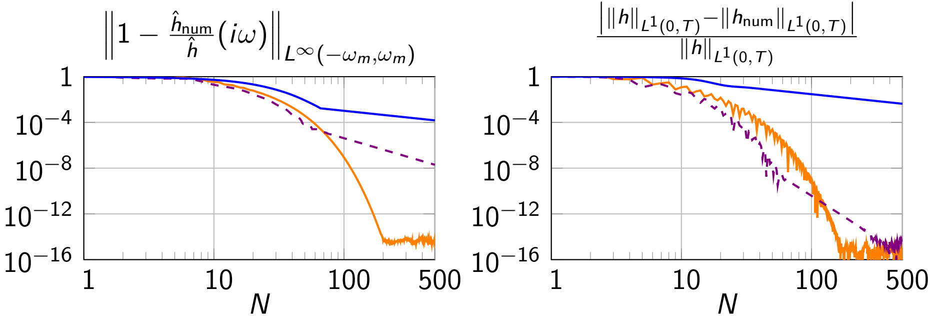

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(- - -) $\beta = \beta_2 \Rightarrow$ Choice from now on

(——) $\beta = \max(\alpha, 1-\alpha)$

Outline

1 Introduction

2 Quadrature-based discretization of diffusive representations

3 Numerical comparisons and applications
- Spectral accuracy of quadrature methods
- Comparison with optimization method
- Application in duct aeroacoustics

4 Conclusion

Eigenvalue approach to stability: overview

Vector-valued fractional delay system: (Monteghetti et al. 2017)

$$\dot{x}(t) = A\,x(t) + B\,x(t-\tau) - G\,d^{1-\alpha}x(t) \quad (\tau \geq 0).$$

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Hyperbolic realisation (PDE)	Parabolic realisation (ODE)
$\theta \in (-\tau, 0)$	$\xi \in (0, \infty)$
$\partial_t \psi = \partial_\theta \psi, \quad \psi(t, \theta = 0) = x(t)$	$\partial_t \varphi = -\xi\varphi + x, \varphi(0, \xi) = u(0)/\xi$
$x(t-\tau) = \psi(t, -\tau)$	$\mathrm{d}^{1-\alpha}x = \int_0^\infty (\cdots)\,\mu_\alpha(\xi)\,\mathrm{d}\xi$

$\Rightarrow$ Abstract Cauchy problem on H

$$\dot{X}(t) = \mathcal{A}\,X(t), \quad \text{with } X := (x, \psi, \varphi).$$

$$\text{Stability} \iff \text{spectrum } \sigma(\mathcal{A})$$

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Discretization

High-order DG-FEM (N_p)	$Q_{\beta,N}$ or optimization (N)

$$\dot{X}_h(t) = \mathcal{A}_h\,X_h(t) \quad \text{with} \quad X_h := (x, \underbrace{\psi_h}_{N_p}, \underbrace{\varphi_h}_{N})$$

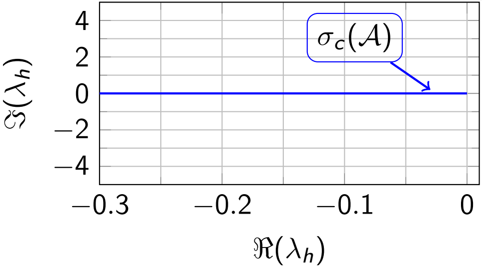
Challenge Ensure $\sigma(\mathcal{A}_h)$ is “meaningful”.

Eigenvalue approach to stability: spectral accuracy (1)

A and B so that $\mathcal{A}$ is asymptotically stable (Monteghetti et al. 2017).

Theoretically

$$S(\mathcal{A}) := \sup_{\lambda \in \sigma(\mathcal{A})} \Re(\lambda) = 0$$



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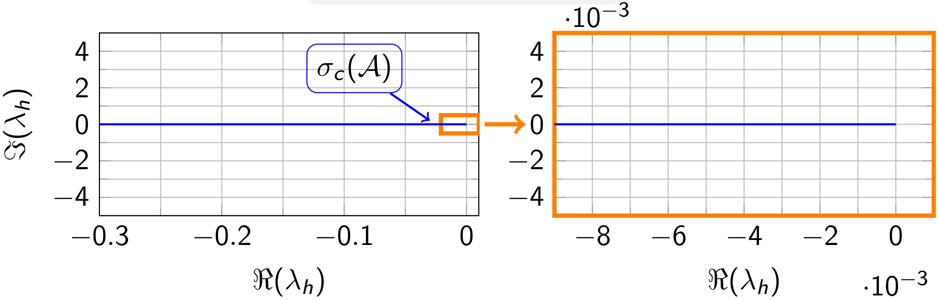

$Q_{\beta,N}$ with $\beta = \beta_2$			
N			
$S(\mathcal{A}_h)$			

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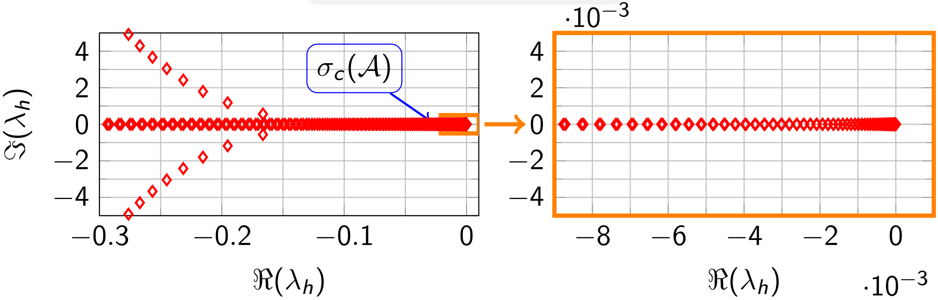

$Q_{\beta,N}$ with $\beta = \beta_2$	(◇)		
N	400		
$S(\mathcal{A}_h)$	-8.1×10^{-11}		

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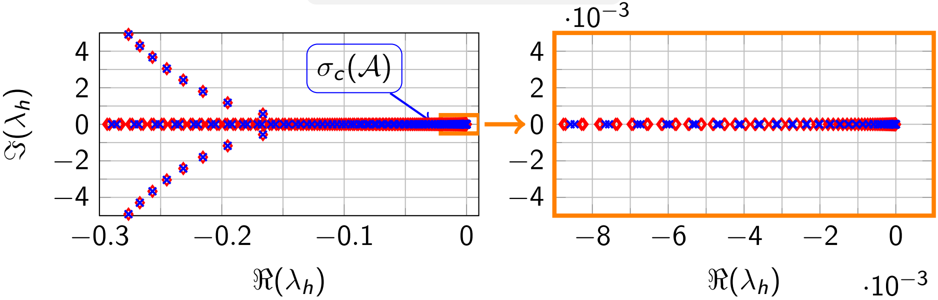

$Q_{\beta,N}$ with $\beta = \beta_2$	(◇)	(×)	
N	400	200	
$S(\mathcal{A}_h)$	-8.1×10^{-11}	-1.3×10^{-9}	

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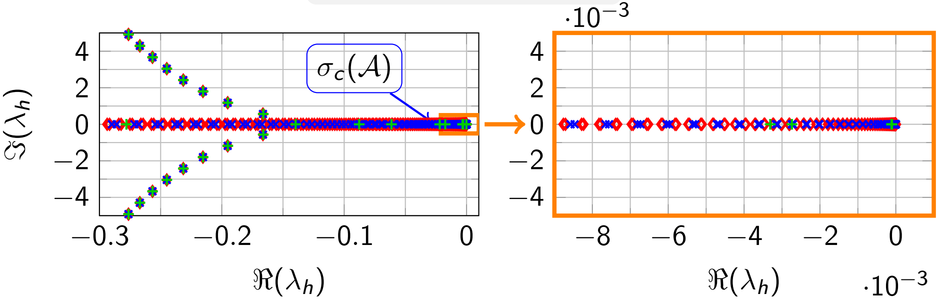


$Q_{\beta,N}$ with $\beta = \beta_2$	(◇)	(×)	(+)
N	400	200	11
$S(\mathcal{A}_h)$	-8.1×10^{-11}	-1.3×10^{-9}	-1×10^{-4}

What about the optimization method?

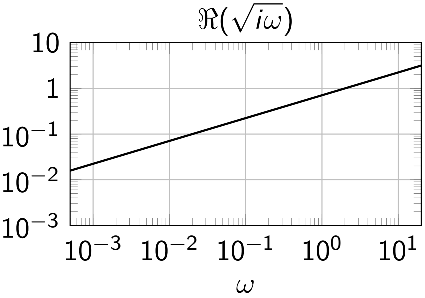
Optimization-based discretization: principle

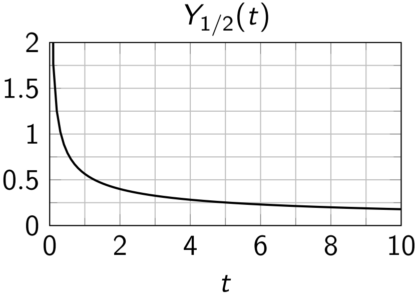
Principle Minimization of (Hélie et al. 2006)

$$(\xi_n, \mu_n) \mapsto \sum_{k=1}^{K} \left| \hat{h}(i\omega_k) - \sum_{n=1}^{N} \frac{\mu_n}{i\omega_k + \xi_n} \right|^2 .$$

Inputs N poles ξ_n in $[\xi_{\min}, \xi_{\max}]$

Illustration





Optimization-based discretization: principle

Principle Minimization of (Hélie et al. 2006)

$$(\xi_n, \mu_n) \mapsto \sum_{k=1}^{K} \left| \hat{h}(i\omega_k) - \sum_{n=1}^{N} \frac{\mu_n}{i\omega_k + \xi_n} \right|^2 .$$

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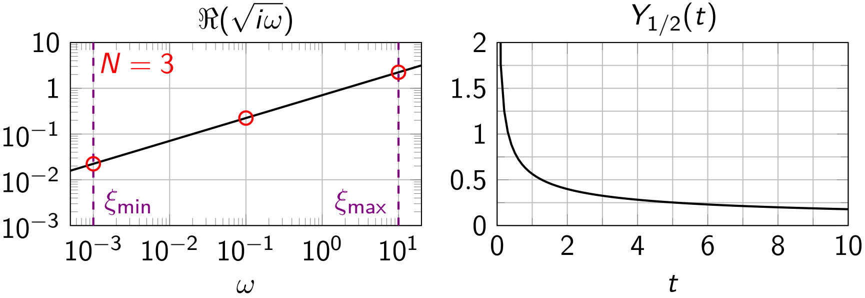

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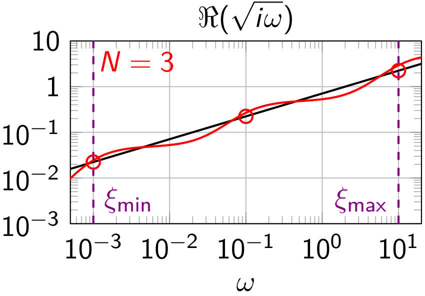

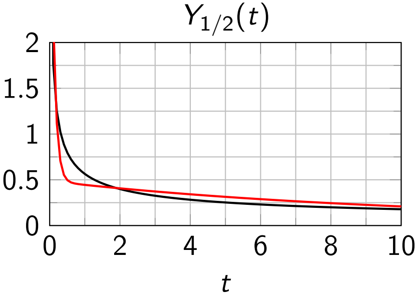

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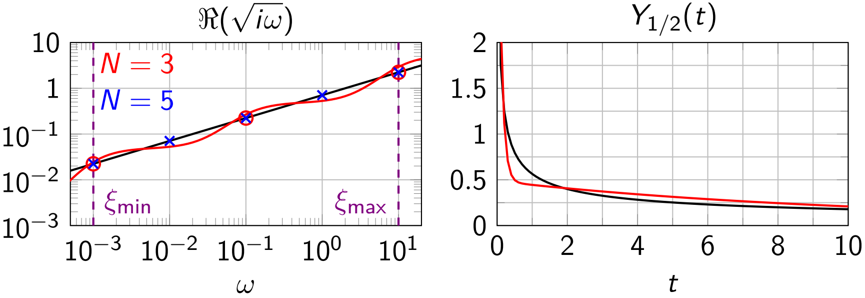

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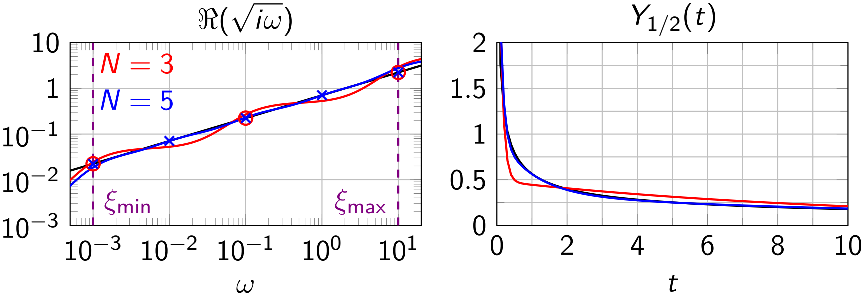
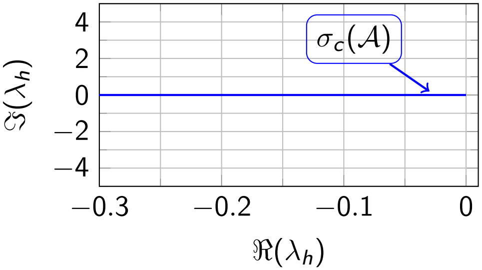


Eigenvalue approach to stability: spectral accuracy (2)

What about the optimization method?



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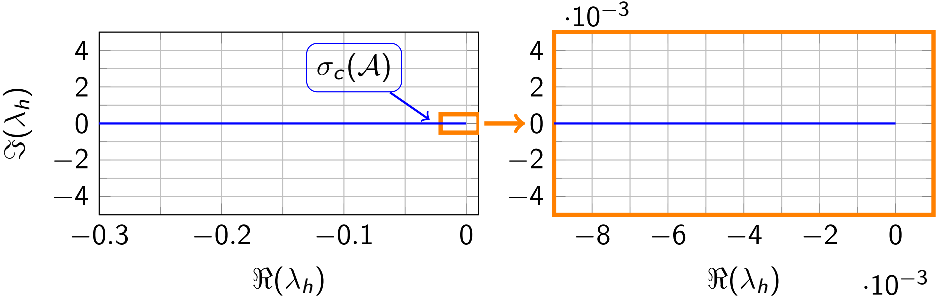

Let $\xi_{\min} = 10^{-16}$.

	$\xi_{\max} = 10^4$		$\xi_{\max} = 10^6$	
N				
$S(\mathcal{A}_h)$				

Eigenvalue approach to stability: spectral accuracy (2)

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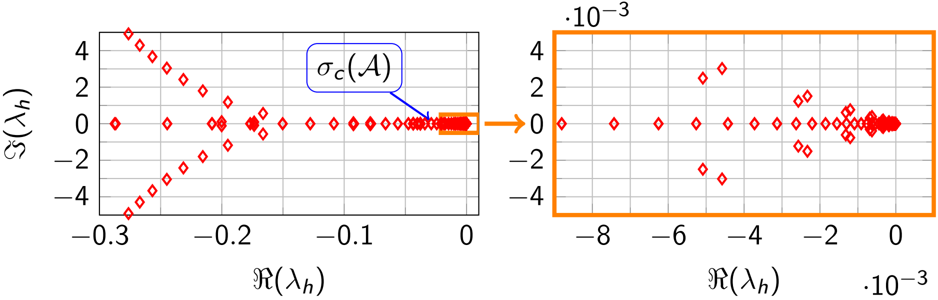

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	$\xi_{\max} = 10^4$		$\xi_{\max} = 10^6$	
	(◇)			
N	400			
$S(\mathcal{A}_h)$	3.4×10^{-13}			

Eigenvalue approach to stability: spectral accuracy (2)

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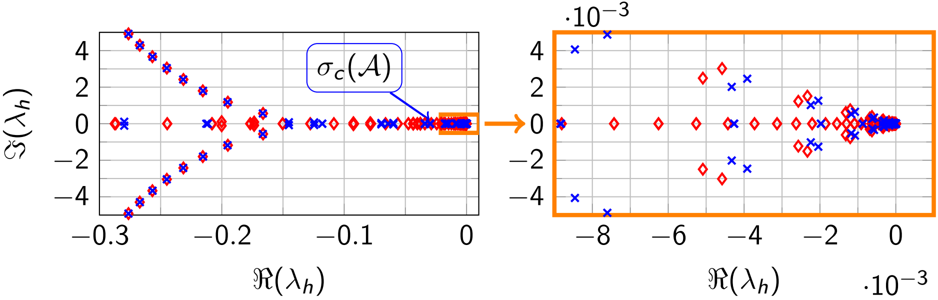
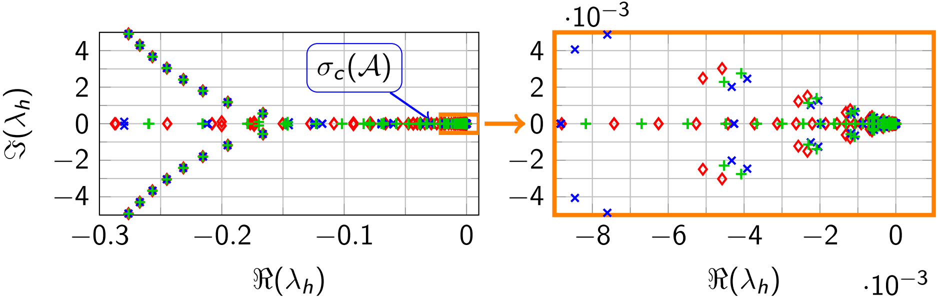


Let $\xi_{\min} = 10^{-16}$.

	$\xi_{\max} = 10^4$		$\xi_{\max} = 10^6$	
	(◊)	(×)		
N	400	200		
$S(\mathcal{A}_h)$	3.4×10^{-13}	3.8×10^{-14}		

Eigenvalue approach to stability: spectral accuracy (2)

What about the optimization method?



Let $\xi_{\min} = 10^{-16}$.

	$\xi_{\max} = 10^4$		$\xi_{\max} = 10^6$	
	(◇)	(×)	(+)	
N	400	200	400	
$S(\mathcal{A}_h)$	3.4×10^{-13}	3.8×10^{-14}	9×10^{-12}	

Eigenvalue approach to stability: spectral accuracy (2)

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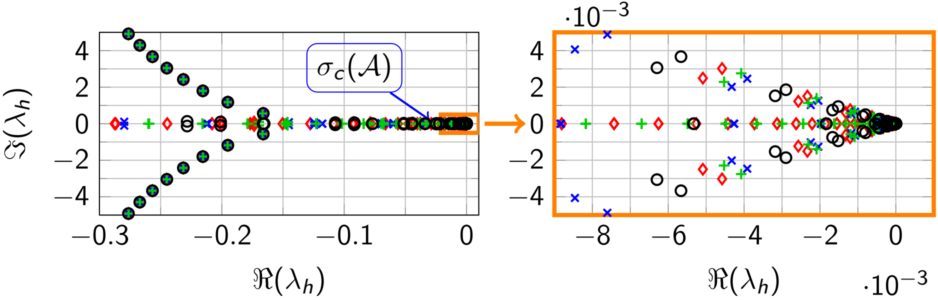


Let $\xi_{\min} = 10^{-16}$.

	$\xi_{\max} = 10^4$		$\xi_{\max} = 10^6$	
	(◊)	(×)	(+)	(○)
N	400	200	400	200
$S(\mathcal{A}_h)$	3.4×10^{-13}	3.8×10^{-14}	9×10^{-12}	1.6×10^{-12}

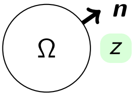
Conclusion $Q_{\beta,N}$ best suited for stability studies.

Duct aeroacoustics: overview

Linearized Euler equations in Ω

$$\partial_t \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} (t, \boldsymbol{x}) + \mathcal{A} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} (t, \boldsymbol{x}) = 0$$

with $p(t,x) = \left[z \underset{t}{\star} \boldsymbol{u} \cdot \boldsymbol{n}(\cdot, x) \right](t)$ on $\partial\Omega$.



Physical impedance model (Monteghetti et al. 2016)

$$\hat{z}_{\text{phys}}(s) = a_0 + a_{1/2}\sqrt{s} + a_1 s + \coth(a_0 + a_{1/2}\sqrt{s} + a_1 s)$$

Duct aeroacoustics: overview

Linearized Euler equations in Ω

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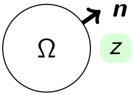

Physical impedance model (Monteghetti et al. 2016)

$$\begin{aligned} \hat{z}_{\text{phys}}(s) &= a_0 + a_{1/2}\sqrt{s} + a_1 s + \coth(a_0 + a_{1/2}\sqrt{s} + a_1 s) \\ &= 1 + a_0 + a_{1/2}\hat{h}_1(s) + e^{-\tau s}\,\hat{h}_2(s). \end{aligned}$$

Duct aeroacoustics: overview

Linearized Euler equations in Ω

$$\partial_t \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} (t, \boldsymbol{x}) + \mathcal{A} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} (t, \boldsymbol{x}) = 0$$

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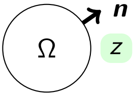

Physical impedance model (Monteghetti et al. 2016)

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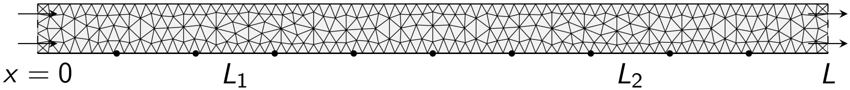
$$= 1 + a_0 + a_{1/2}\hat{h}_1(s) + e^{-\tau s}\hat{h}_2(s).$$

Discretization process $h_i \simeq h_{i,\text{num}}$:

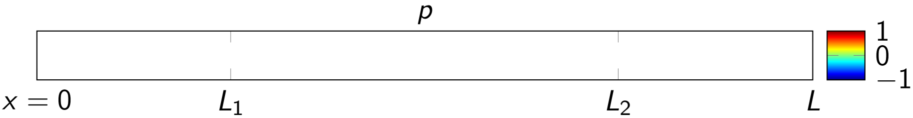
1. Quadrature or optimization $\Rightarrow$ Initial guess for $h_{i,\text{num}} \Rightarrow z_{\text{num}}$
2. Nonlinear least squares against experimental data

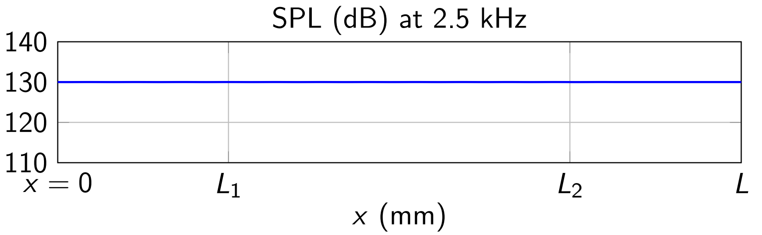
$$\left\| \hat{z}_{\text{num}}(i\omega) - \hat{z}_{\text{exp}}(i\omega) \right\|$$

Duct aeroacoustics: application



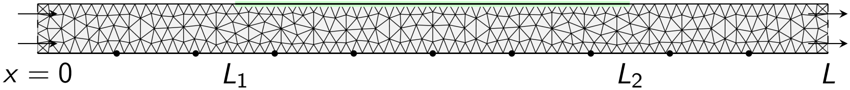
ArtimonDG(4). 552 triangles. CFL = 0.85. LSERK (8,4) (Toulorge et al. 2012)





Duct aeroacoustics: application

$\hat{\beta}_a(s)$ (liner CT57 – NASA Langley)



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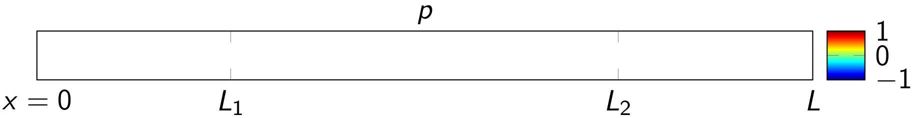

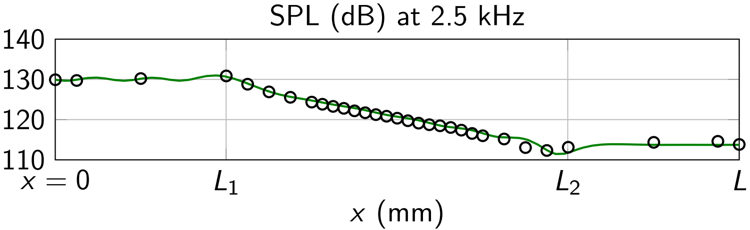

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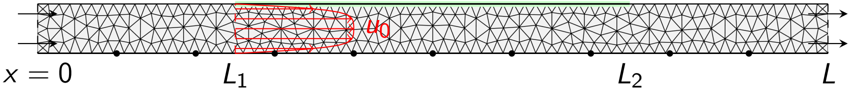

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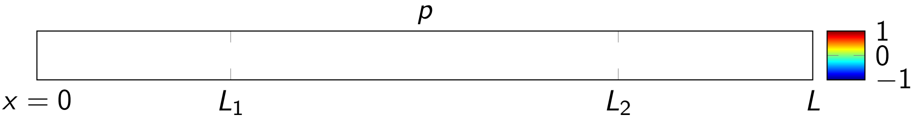

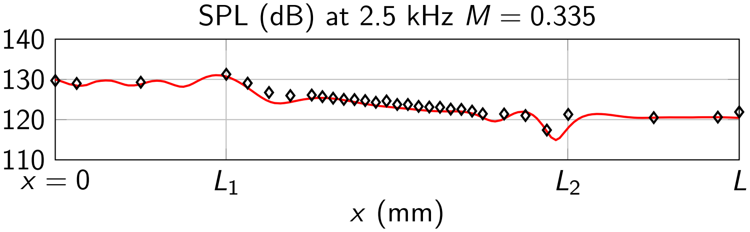

Outline

Conclusion

$Q_{\beta,N}$ method

- ✔ One parameter $\beta(\alpha)$
- ✔ No spectral pollution
- ✘ $\xi_{\max} \propto N^{\frac{2}{\beta}}$

Optimization-based method

- ✘ At least three parameters
- ✘ Polluted spectrum
- ✔ Control of $\xi_{\max}$
- ✔ Suited for wave propagation

Perspectives

- Applications to fractional PDE (e.g. fractional Schrödinger (Garrappa et al. 2015))
- Extension to diffusive operators with singular or sharply-varying weight μ (e.g. Webster-Lokshin, Cavity impedance)
- Enhancement of the optimization method

Table of contents

▸ Main TOC ▸ Additional slides TOC

Quadrature-based diffusive representation
of the fractional derivative with applications
in aeroacoustics and eigenvalue methods for stability

Thanks for your attention. Any questions?

Contact: `florian.monteghetti@onera.fr`

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