# Stability of Linear Fractional Differential Equations with Delays

## A coupled Parabolic-Hyperbolic PDEs formulation

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# Outline

# Motivation: fractional delay systems in aeroacoustics

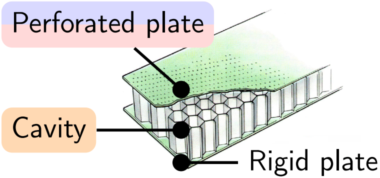
**Context:** Noise regulations ⇒ research into sound absorption.

## Modelling of locally-reacting sound absorbing material

Passive LTI system:

$$p(t,x) = \left[z \underset{t}{\star} \boldsymbol{u} \cdot \boldsymbol{n}(\cdot,x)\right](t)$$

with kernel $z \in \mathcal{D}'_+(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$.



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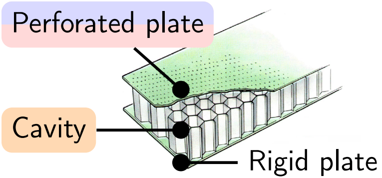


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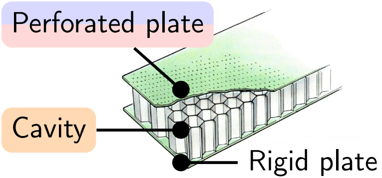


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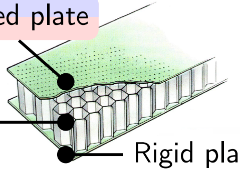


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$\Rightarrow$ Spatial discretisation yields fractional delay equation ($x \in \mathbb{R}^n$):

$$M \cdot \dot{x}(t) + K \cdot x(t) = F_1 \cdot d^{1/2}x(t) + F_2 \cdot x(t-\tau).$$

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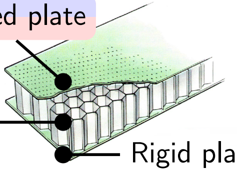


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**Objective:** use parabolic-hyperbolic realisations to study stability.

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1. Introduction
2. Coupled PDEs formulation: stability results
   - Existing stability results
   - Scalar “toy” model
   - Vector-valued model
3. An eigenvalue approach to stability
4. Conclusion

# Existing stability results

**Time-delay system of retarded type**

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t - \tau_i) \Leftrightarrow \dot{X} = \mathcal{A} X$$

- Roots of characteristic equation $\det \Delta(\lambda) = 0 \Leftrightarrow \lambda \in \sigma_p(\mathcal{A})$.
  (Michiels and Niculescu 2014, Chap. 1) (Curtain and Zwart 1995, § 2.4)
- Lyapunov-Krasovkii equivalence theorem.
  (Fridman 2014, Chap. 3) (Briat 2014, Thm. 5.2.9)

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## Fractional differential equation

- BIBO & asymptotic stability with commensurate fractional powers. (Matignon 1996)

## Fractional delay differential equation

- BIBO stability with commensurate delays.
  (Bonnet and Partington 2002)
- Asymptotic stability with non-commensurate delays.
  (Deng, Li and Lü 2007)

# Toy model: Laplace technique

**Objective:** delay-independent stability of

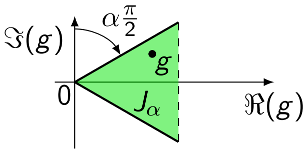
$$\dot{x}(t) = a\,x(t) + b\,x(t-\tau) - g\,d_C^{\alpha}x(t) \quad \text{for } t > \tau,\ \alpha \in (0,1)$$
$$x(t) := x^0(t) \quad \text{for } t \in [0,\tau].$$

**Theorem. Toy model stability**

Under the following algebraic condition:

$$\Re(a) < -|b| \leq 0 \quad \text{and} \quad g \in J_\alpha,$$

toy model is delay-independent stable.



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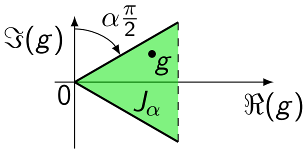


**Proof (Sketch).** Expression of $\hat{x}(s)$ has 5 terms:

$$\hat{x}(s) = g\,x^0(0)\,\hat{h}_c(s) + g\,x^0(\tau)\,\hat{h}_d(s) + x^0(\tau)\,\hat{h}_e(s) + \widehat{x^0}\,\hat{h}_a(s) + g\mathcal{L}[\mathrm{d}_C^\alpha x^0\,\mathbb{1}_{[0,\tau]}]\,\hat{h}_b(s).$$

1. $\hat{h}_{c,d,e}(s)$: final-value theorem.
2. $\hat{h}_{a,b}(s)$: Callier-Desoer $\hat{\mathcal{A}}(0)$ class and dominated convergence.

# Toy model: coupled PDEs formulation (1)

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**Proof.** Let $E_x := \frac{1}{2}|x|^2$. Decay rate along trajectories is

$$\dot{E}_x = 2\,\Re(a)\,E_x + \Re\Big[\,\overline{x}\,(b\,x_\tau - g\,d_C^{\alpha}x(t))\Big], \tag{1}$$

whose sign is *a priori* indefinite.

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whose sign is *a priori* indefinite.

However, energy decay can be proven using suitable realisations.

1. Hyperbolic for $x_\tau(t)$: transport equation.
2. Parabolic for $D_{RL}^\alpha\,x(t)$: heat equation.
3. Extended energy $\mathcal{E}$ $\Rightarrow$ sufficient condition for decay.

# Toy model: coupled PDEs formulation (2)

**Hyperbolic realisation $z \in (0, \ell)$** Transport PDE.

(Engel and Nagel 2000, § VI.6) (Curtain and Zwart 1995, § 2.4)

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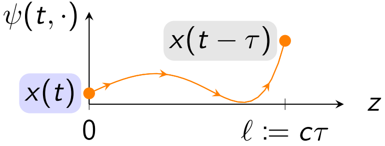


$$\begin{cases} \partial_t \psi(t,z) = -c\,\partial_z \psi(t,z) \\ \psi(t, z=0) := x(t) \\ x(t-\tau) = \psi(t, z=\ell) \end{cases}$$

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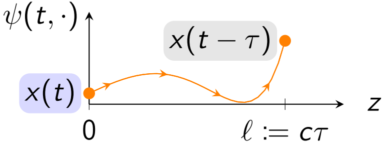


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Natural energy: $$E_\psi(t) := \frac{1}{2} \int_0^\ell |\psi(t,z)|^2 \, \mathrm{d}z \,.$$

Energy balance reflects lossless transport:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_\psi(t) &= -c \int_0^\ell \Re(\partial_z \psi(t,z)\, \overline{\psi}(t,z))\, \mathrm{d}z \\ &= -\frac{c}{2} \left[\, |\psi(t,z)|^2 \,\right]_0^\ell \\ &= \frac{c}{2} \left( |x(t)|^2 - |x(t-\tau)|^2 \right). \end{aligned}$$

# Toy model: coupled PDEs formulation (3)

**Parabolic realisation** $\xi \in (0, \infty)$ (Parabolic) ODE.

(Staffans 1994) (Montseny 1998) (Matignon 2009) (Hélie and Matignon 2006a)

$$\begin{cases} \partial_t \varphi(\xi, t) = -\xi\, \varphi(\xi, t) + u(t), \;\; \varphi(\xi, 0) = 0\,, \\ D^{\alpha}_{\mathrm{RL}}\, x(t) = \displaystyle\int_0^{\infty} \mu_{1-\alpha}(\xi)\, [u(t) - \xi\, \varphi(\xi, t)]\, \mathrm{d}\xi\,. \end{cases}$$

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Natural energy: $E_{\varphi}(t) := \frac{1}{2} \int_0^{\infty} \xi\, |\varphi(\xi, t)|^2\, \mu_{1-\alpha}(\xi) \mathrm{d}\xi.$

Energy balance (expresses dissipativity of $D^{\alpha}_{\mathrm{RL}}$):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_{\varphi}(t) &= \Re(\overline{x}\, D^{\alpha}_{\mathrm{RL}}\, x) - \int_0^{\infty} |x - \xi\, \varphi(\xi, \cdot)|^2\, \mu_{1-\alpha}(\xi)\, \mathrm{d}\xi\,. \\ &\leq \Re(\overline{x}\, D^{\alpha}_{\mathrm{RL}}\, x). \end{aligned}$$

# Toy model: coupled PDEs formulation (4)

**Extended energy** $\mathcal{E}_k := E_x(t) + k\, E_\psi(t) + g\, E_\varphi(t)$ ,
with $k > 0$ unknown.

- Parabolic realisation: cross terms $g\,\Re(\overline{x}\, D^{\alpha}_{\mathrm{RL}} x)$ cancel out

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- Hyperbolic realisation leads to

$$\dot{\mathcal{E}}_k \leq -X^H \Sigma_k X$$

where $X := (x, x_\tau)^\intercal$ and

$$\Sigma_k := -\begin{pmatrix} \Re(a) + k\frac{c}{2} & \frac{b}{2} \\ \frac{\overline{b}}{2} & -k\frac{c}{2} \end{pmatrix} \overset{?}{>} 0.$$

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- Eigenvalues: $\lambda_2(k) > \lambda_1(k) = -\frac{\Re(a)+\sqrt{P(k)}}{2}$, with $P(k) > 0$.
- Least stringent condition:

$$\min_{k>0} \lambda_1(k) = -\frac{\Re(a) + |b|}{2} \quad \text{for} \quad k^* = -\frac{\Re(a)}{c}.$$

$$\lambda_1 > 0 \iff \Re(a) < -|b|$$

□

# Vector-valued model

Vector-valued fractional system with delay:

$$\dot{x}(t) = Ax(t) + B\,x(t-\tau) - G\,d_C^\alpha x(t) \quad \text{for } t > \tau,\ \alpha \in (0,1)$$
$$x(t) := x^0(t) \quad \text{for } t \in [0,\tau],$$

with $x(t) \in \mathbb{R}^n$.

**Theorem. Stability.**

Let $G$ be a diagonalisable matrix with eigenvalues $(g_1, \cdots, g_n) \geq 0$. Under the algebraic condition

$$\max_{a \in \sigma(A)} \Re(a) < -\sqrt{\max_{b \in \sigma(B^H B)} |b|} \leq 0,$$

the system with $x^0(0) = 0$ is delay-independent stable.

**Proof.** Similar in spirit to toy model, with extended energy

$$\mathcal{E}(t) := \sum_{i \in [\![1,n]\!]} E_{x_i}(t) + k\,E_{\psi_i}(t) + g_i\,E_{\varphi_i}(t).$$

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3. An eigenvalue approach to stability
   - State of the art
   - Eigenvalue approach to stability
4. Conclusion

# Numerical methods for stability: state of the art

**Time-delay system of retarded type**

- Design an approximate Lyapunov-Krasovkii functional, and formulate a numerically-tractable LMI.

  (Seuret, Gouaisbaut and Ariba 2015) (Baudouin, Seuret and Safi 2016)

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  (Seuret, Gouaisbaut and Ariba 2015) (Baudouin, Seuret and Safi 2016)
- Study of characteristic roots. (Michiels and Niculescu 2014, § 2)
  - Count unstable roots. (Li, Niculescu and Cela 2015)
  - Locate unstable roots : eigenvalue approach.
    - Spectrum of operator semigroup $e^{t\mathcal{A}}$.
      DDE-BIFTOOL (Engelborghs, Luzyanina and Roose 2002)
    - Spectrum of generator $\mathcal{A}$ using hyperbolic realisation.
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## Fractional delay differential equation

- YALTA. (Fioravanti et al. 2012) (Avanessoff, Fioravanti and Bonnet 2013)
- Count unstable roots. (Zhang et al. 2016)

$\Rightarrow$ Eigenvalue approach using parabolic-hyperbolic realisation?

# Eigenvalue approach to stability: overview

Vector-valued fractional delay system:

$$\dot{x}(t) = A\,x(t) + B\,x(t-\tau) - G\,d_C^{\alpha}x(t-\tau_\alpha) \quad (\tau_\alpha \geq 0).$$

| Hyperbolic realisation (PDE) | Parabolic realisation (ODE) |
|---|---|
| $z \in (0,1)$ | $\xi \in (0,\infty)$ |
| $\partial_t \psi_h = -\tau^{-1}\partial_z \psi_h$ <br> $\psi_h(0) = x$ <br> $x(t-\cdot) = \psi_h(z=1)$ | $\partial_t \varphi_h = -\xi \varphi_h + x$ <br> $d_C^{\alpha}x = \sum_{k \in [\![1,N_\xi]\!]} \mu_k\,\varphi_h(\xi_k)$ |
| $\Rightarrow$ High-order discretisation | $\Rightarrow$ Quadrature or optimisation |

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| $\Rightarrow$ High-order discretisation | $\Rightarrow$ Quadrature or optimisation |

$\Rightarrow$ Cauchy problem on $\mathbb{C}^n$:

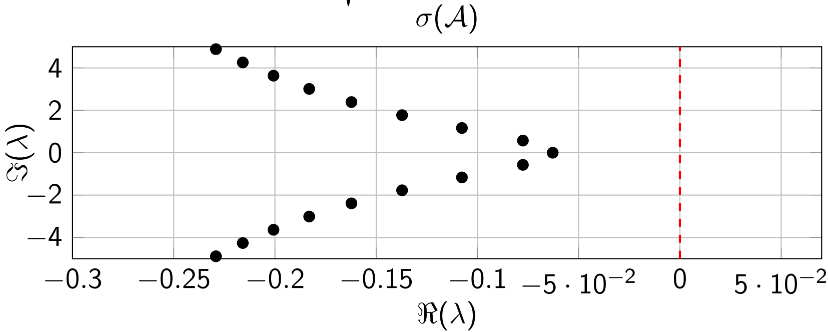
$$\dot{X}_h(t) = \mathcal{A}_h\,X_h(t), \quad \text{with } X_h := (x, \psi_h, \varphi_h).$$

Challenge: ensure $\sigma(\mathcal{A}_h)$ is "meaningful".

# Numerical experiment: spectral structure

**Case 1:** $x(t) \in \mathbb{R}^2$, $\dot{x}(t) = A \cdot x(t) + B \cdot x(t-\tau) - g\, I_2 \cdot d_C^{1/2} x(t)$, with

$$\max_{a \in \sigma(A)} \Re(a) < -\sqrt{\max_{b \in \sigma(B^H B)} |b|} \leq 0 \quad \text{verified.}$$

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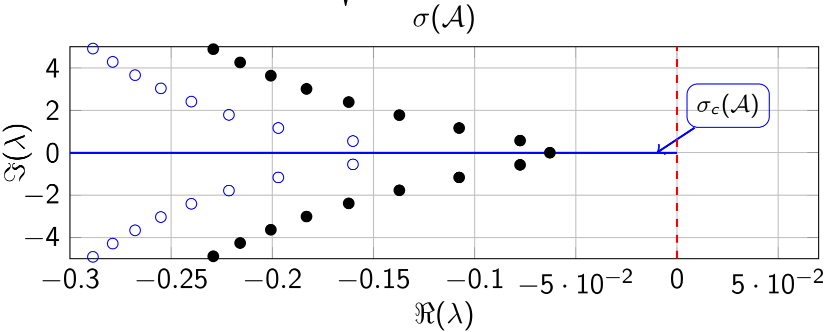


**Pure delay** $g = 0$ (•) $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ (discrete)

**Fractional derivative** $g \neq 0 \Rightarrow \sigma_c(\mathcal{A}) \neq \emptyset$ (essential)

(○) $g = +2 > 0 \Rightarrow$ stable

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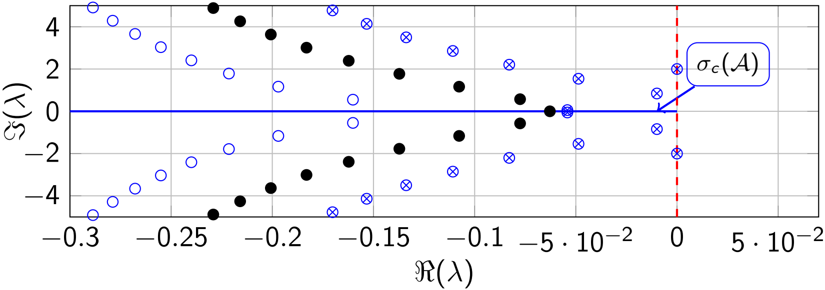
$\sigma(\mathcal{A})$




**Pure delay** $g = 0$ (•) $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ (discrete)

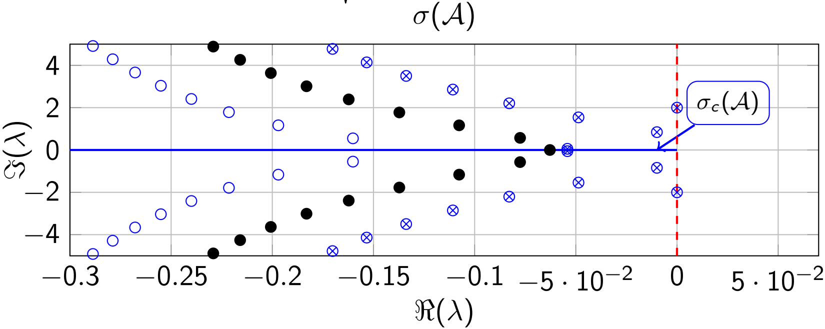
**Fractional derivative** $g \neq 0 \Rightarrow \sigma_c(\mathcal{A}) \neq \emptyset$ (essential)

(○) $g = +2 > 0 \Rightarrow$ stable (⊗) $g = -2 < 0 \Rightarrow$ unstable

# Numerical experiment: spectral structure

**Case 1:** $x(t) \in \mathbb{R}^2$, $\dot{x}(t) = A \cdot x(t) + B \cdot x(t-\tau) - g\, I_2 \cdot d_C^{1/2} x(t)$, with

$$\max_{a \in \sigma(A)} \Re(a) < -\sqrt{\max_{b \in \sigma(B^H B)} |b|} \leq 0 \quad \text{verified.}$$

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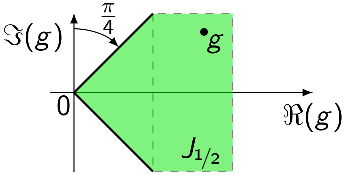
(∘) $g = +2 > 0 \Rightarrow$ stable (⊗) $g = -2 < 0 \Rightarrow$ unstable

What about $g \in \mathbb{C}$?

# Numerical experiment: delay-dependent stability

**Case 2:** Scalar model $\dot{x}(t) = -x(t) + \frac{1}{2}x(t-\tau) - g\, d_C^{1/2} x(t)$.
For delay-independent stability, $g \in J_{1/2}$.



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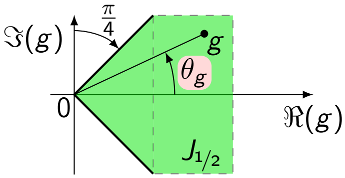
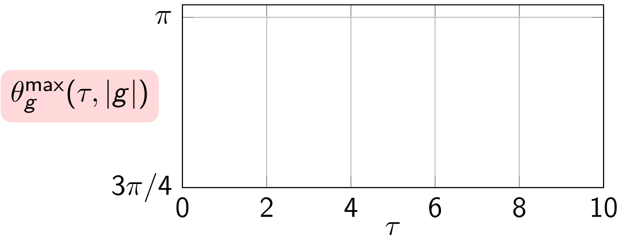



What about the delay-dependent stability region?



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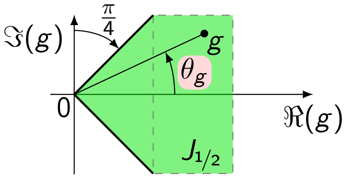


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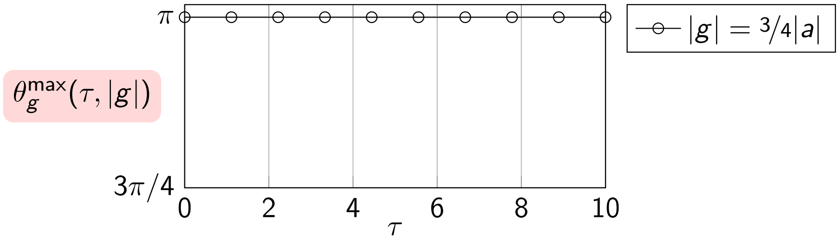

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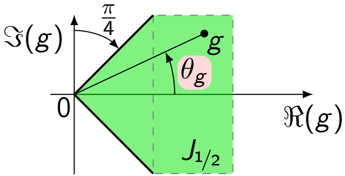


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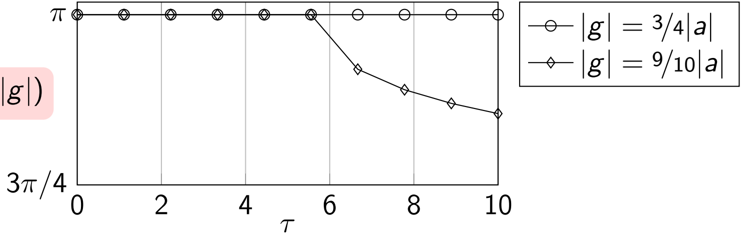

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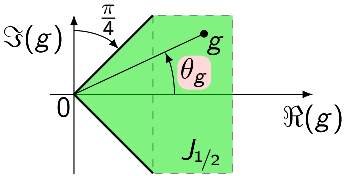


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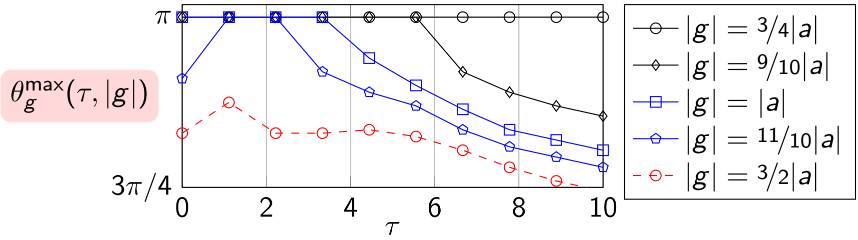
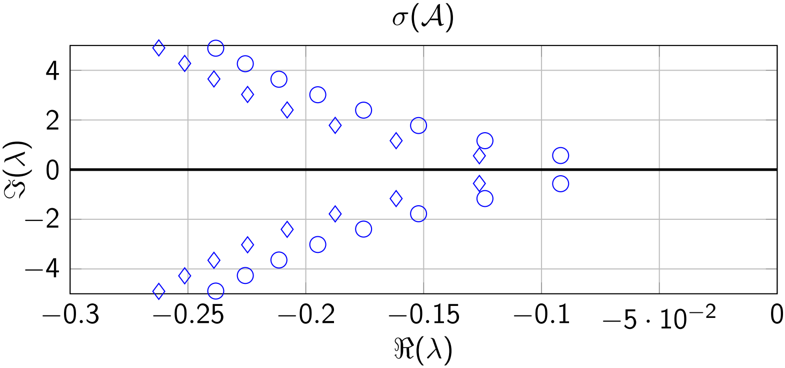


# Numerical experiment: composition (exploratory)

**Case 3:** Scalar model $\dot{x}(t) = -x(t) + \frac{1}{2}x(t-\tau) - g\, d_C^{1/2} x(t - \tau_\alpha)$.




## Effect of delaying the fractional derivative

$g = |a|/4$.

1. $\tau_\alpha = 0$ (○).

$g = |a|$.

1. $\tau_\alpha = 0$ (◇).

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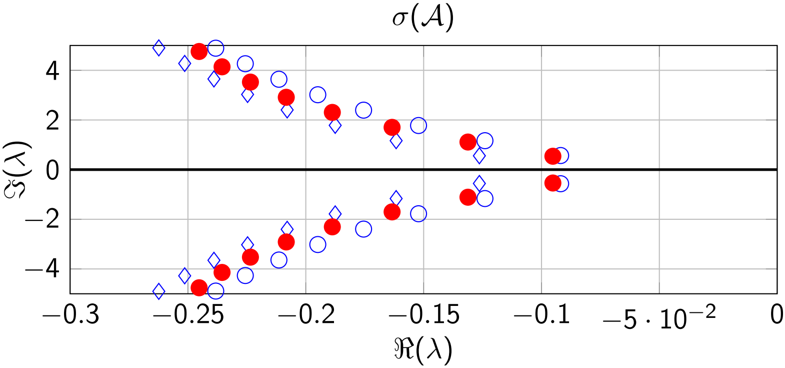


## Effect of delaying the fractional derivative

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2. $\tau_\alpha = \tau$ (●).

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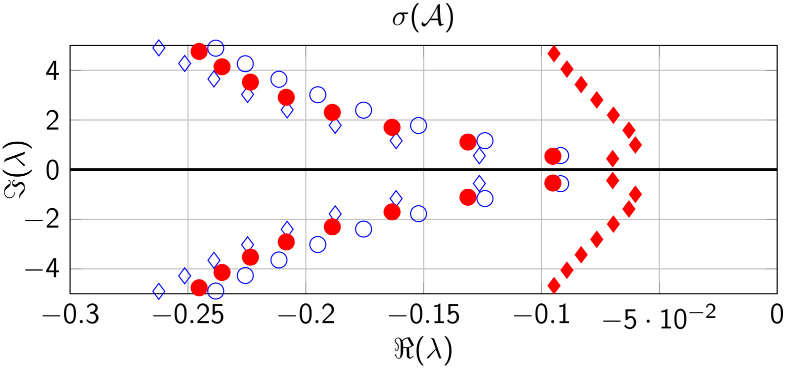


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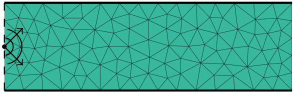
# Application in acoustics

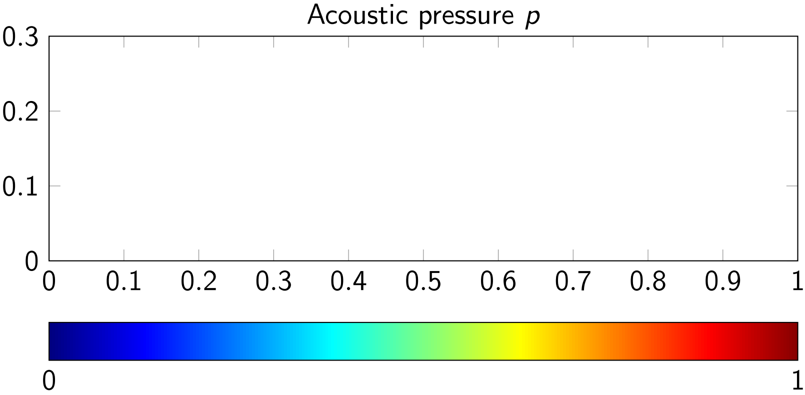
**Computational case** Infinite 2D duct.

DG: $N = 4$. Mesh: $N_K = 188$.

Time-integration: CFL $= 0.5$. (LSERK (8,4) (Toulorge and Desmet 2012))



$\hat{z}(s, x) = \infty$ (Rigid Wall)



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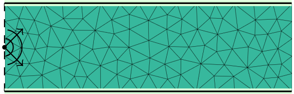
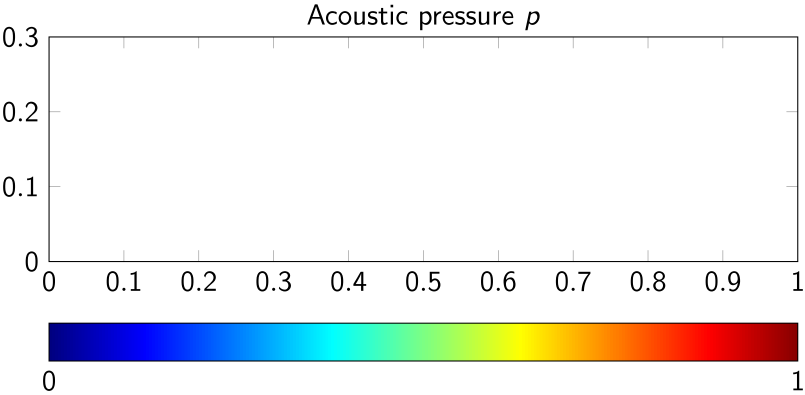


$$\hat{z}(s) = a + a\sqrt{s} + \frac{a}{2}e^{-s\tau} \quad \text{(Soft Wall)}$$



# Outline

# Conclusion

## Takeaways

- Parabolic-Hyperbolic PDE realisations $\Rightarrow$ time-local coupled system $(x, \varphi, \psi)$
- Natural extended energy $\mathcal{E} = E_x + E_\varphi + k\,E_\psi$

  $\Rightarrow$ sufficient asymptotic stability condition

  $\Rightarrow$ eigenvalue approach to stability
- Application to aeroacoustics

## Perspectives

- Multiple delay case
- Semigroup formulation
- Composition: $D^{\alpha}_{\mathrm{RL}} x(t-\tau)$?
- Theoretical study of eigenvalue approach

# Conclusion



▸ Appendix

Thanks for your attention. Any questions?

(Contact: `florian.monteghetti@onera.fr`)

# References I

Avanessoff, D., A. Fioravanti and C. Bonnet (2013). "YALTA: a Matlab toolbox for the $H_\infty$-stability analysis of classical and fractional systems with commensurate delays". In: *IFAC Proceedings Volumes* 46.2, pp. 839–844. ISSN: 1474-6670 (cit. on pp. 27–29).

Baudouin, L., A. Seuret and M. Safi (2016). "Stability analysis of a system coupled to a transport equation using integral inequalities". In: *IFAC-PapersOnLine* 49.8, pp. 92–97. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2016.07.424 (cit. on pp. 27–29).

Bonnet, C. and J. R. Partington (2002). "Analysis of fractional delay systems of retarded and neutral type". In: *Automatica* 38.7, pp. 1133–1138. ISSN: 0005-1098 (cit. on pp. 10, 11).

Breda, D., S. Maset and R. Vermiglio (2005). "Pseudospectral Differencing Methods for Characteristic Roots of Delay Differential Equations". In: *SIAM Journal on Scientific Computing* 27.2, pp. 482–495. DOI: 10.1137/030601600 (cit. on pp. 27–29).

# References II

Briat, C. (2014). *Linear Parameter-Varying and Time-Delay Systems*. Berlin: Springer (cit. on pp. 10, 11).

Curtain, R. F. and H. Zwart (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer (cit. on pp. 10, 11, 18, 19, 60).

Deng, W., C. Li and J. Lü (2007). "Stability analysis of linear fractional differential system with multiple time delays". In: *Nonlinear Dynamics* 48.4, pp. 409–416. ISSN: 1573-269X. DOI: 10.1007/s11071-006-9094-0 (cit. on pp. 10, 11).

Engel, K.-J. and R. Nagel (2000). *One-parameter semigroups for linear evolution equations*. New York: Springer-Verlag (cit. on pp. 18, 19).

Engelborghs, K., T. Luzyanina and D. Roose (2002). "Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL". In: *ACM Transactions on Mathematical Software (TOMS)* 28.1, pp. 1–21 (cit. on pp. 27–29).

# References III


Fioravanti, A. R. et al. (2012). "A numerical method for stability windows and unstable root-locus calculation for linear fractional time-delay systems". In: *Automatica* 48.11, pp. 2824–2830. ISSN: 0005-1098. DOI: 10.1016/j.automatica.2012.04.009 (cit. on pp. 27–29).

Fridman, E. (2014). *Introduction to Time-Delay Systems*. Basel: Birkhäuser (cit. on pp. 10, 11).

Heleschewitz, D. (2000). "Analyse et simulation de systemes différentiels fractionnaires et pseudo-différentiels sous representation diffusive". PhD thesis. ENST Paris (cit. on p. 66).

Hélie, T. and D. Matignon (2006a). "Diffusive representations for the analysis and simulation of flared acoustic pipes with visco-thermal losses". In: *Mathematical Models and Methods in Applied Sciences* 16.04, pp. 503–536. DOI: 10.1142/S0218202506001248 (cit. on pp. 20, 21).

# References IV


Hélie, T. and D. Matignon (2006b). "Representations with poles and cuts for the time-domain simulation of fractional systems and irrational transfer functions". In: *Signal Processing* 86.10, pp. 2516–2528. DOI: 10.1016/j.sigpro.2006.02.017 (cit. on p. 66).

Li, X.-G., S.-I. Niculescu and A. Cela (2015). *Analytic curve frequency-sweeping stability tests for systems with commensurate delays*. Springer (cit. on pp. 27–29).

Matignon, D. and H. J. Zwart (in revision). "Standard diffusive systems as well-posed linear systems". In: *Int. J. Control* (cit. on p. 63).

Matignon, D. (1996). "Stability results for fractional differential equations with applications to control processing". In: *IMACS/IEEE-SMC conference on Computational Engineering in Systems Application*. Vol. 2. Lille, France, pp. 963–968 (cit. on pp. 10, 11).

# References V

Matignon, D. (2009). "An introduction to fractional calculus". In: *Scaling, Fractals and Wavelets, vol. 1*. Ed. by P. Abry, P. Gonçalvès and J. Levy-Vehel. Digital signal and image processing series. ISTE - Wiley, pp. 237–277 (cit. on pp. 20, 21).

Michiels, W. and S.-I. Niculescu (2014). *Stability, Control, and Computation for Time-Delay Systems*. 2nd. Philadelphia: SIAM (cit. on pp. 10, 11, 18, 19, 27–29).

Monteghetti, F. et al. (2016). "Design of broadband time-domain impedance boundary conditions using the oscillatory-diffusive representation of acoustical models". In: *The Journal of the Acoustical Society of America* 140.3, pp. 1663–1674. DOI: 10.1121/1.4962277 (cit. on pp. 3–7).

Montseny, G. (1998). "Diffusive representation of pseudo-differential time-operators". In: *ESAIM: Proceedings*. Vol. 5. EDP Sciences, pp. 159–175. DOI: 10.1051/proc:1998005 (cit. on pp. 20, 21).

# References VI

Seuret, A., F. Gouaisbaut and Y. Ariba (2015). "Complete quadratic Lyapunov functionals for distributed delay systems". In: *Automatica* 62, pp. 168–176. ISSN: 0005-1098. DOI: 10.1016/j.automatica.2015.09.030 (cit. on pp. 27–29).

Shampine, L. (2008). "Vectorized adaptive quadrature in MATLAB". In: *J. Comput. Appl. Math.* 211.2, pp. 131–140. ISSN: 0377-0427 (cit. on p. 66).

Staffans, O. J. (1994). "Well-posedness and stabilizability of a viscoelastic equation in energy space". In: *Trans. Amer. Math. Soc.* 345, pp. 527–575 (cit. on pp. 20, 21).

Toulorge, T. and W. Desmet (2012). "Optimal Runge-Kutta schemes for discontinuous Galerkin space discretizations applied to wave propagation problems". In: *Journal of Computational Physics* 231.4, pp. 2067–2091. ISSN: 0021-9991. DOI: 10.1016/j.jcp.2011.11.024 (cit. on pp. 44, 45).

# References VII

Zhang, L. et al. (2016). "Complete stability of linear fractional order time delay systems: A unified frequency-sweeping approach". In: *35th Chinese Control Conference (CCC)*. Chengdu, pp. 1605–1609 (cit. on pp. 27–29).