Asymptotic stability of LEE with long-memory impedance boundary condition

Theoretical and numerical considerations

WAVES 2017, Minneapolis — 19 May 2017

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Outline

Introduction & Objectives

Context. Noise regulations $\Rightarrow$ research effort into sound generation / absorption / propagation.

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PDE. Linearised Euler equations on $\Omega \subset \mathbb{R}^n$ with base flow $\boldsymbol{u}_0$

$$\partial_t \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} + \underbrace{\begin{bmatrix} \nabla p \\ \nabla \cdot \boldsymbol{u} \end{bmatrix}}_{\text{acoustics}} + \underbrace{\begin{bmatrix} (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_0 + p\,(\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 \\ \boldsymbol{u}_0 \cdot \nabla p + p\,\nabla \cdot \boldsymbol{u}_0 \end{bmatrix}}_{\text{aero-acoustics}} = \mathbf{0}$$

with an *impedance boundary condition*

$$p(x,t) = Q(\boldsymbol{u}(x,t) \cdot \boldsymbol{n}) \quad x \in \partial\Omega,$$

where Q is an operator that "dissipates" energy.

Objectives

Theory — Well-posedness & stability

Numerics — Numerical scheme suitable for

- high-order time-domain simulations
- stability studies

What class of operator for Q?

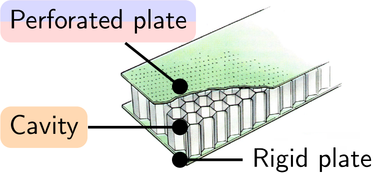
Overview of impedance models

Modelling of locally-reacting sound absorbing material

Q: linear time-invariant operator

$$p(t,x) = \left[z \underset{t}{\star} \boldsymbol{u} \cdot \boldsymbol{n}(\cdot,x)\right](t)$$

with kernel $z \in \mathcal{D}'_+(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$.



Key components of z: (Monteghetti, Matignon et al. 2016, JASA)

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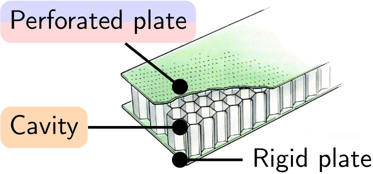

Key components of z: (Monteghetti, Matignon et al. 2016, JASA)

1. Acoustic resonator (Helmholtz then Rayleigh) (Lamb 1910, p.260)

$$p(t) = k \int_0^t u(\eta)\,\mathrm{d}\eta + m\,\dot{u}(t) \quad \Rightarrow \quad \hat{z}(s) = \frac{k}{s} + m\,s \quad (\Re(s) > 0).$$

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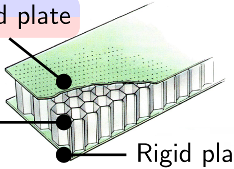

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2. Viscous losses: memoryless and long-memory damping
   Wave reflection: delay

$$p(t) = \cdots + a_0\,u(t) + a_{1/2} \int_0^t \frac{1}{\sqrt{\pi(t-\eta)}} \star \dot{u}(\eta)\,\mathrm{d}\eta + a_\tau\,u(t-\tau)$$

$$\Rightarrow \hat{z}(s) = \cdots + a_0 + a_{1/2}\sqrt{s} + a_\tau\,e^{-s\tau}$$

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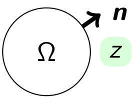
Objectives & Strategy

Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{u}\\ p\end{bmatrix} = \mathcal{A}_{\mathrm{ac}}\begin{bmatrix}\boldsymbol{u}\\ p\end{bmatrix} = \begin{bmatrix}-\nabla p\\ -\nabla\cdot u\end{bmatrix}$$

with $p = z \star \boldsymbol{u}\cdot\boldsymbol{n}$ on $\partial\Omega$.



Motivation: focus on impedance models z (basis for numerical method).

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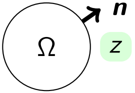

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What are we interested in ?

- Well-posedness $\exists C > 0 : \forall t > 0,\ \|x(t)\|_{\mathcal{H}} \leq C\,\|x_0\|_{\mathcal{H}}$
- Stability (Luo, Guo and Morgül 2012, Def. 3.1)

  Asymptotic $\forall x_0,\ \|x(t)\|_{\mathcal{H}} \to 0$ for $t \to \infty$

  Exponential $\exists C, \omega > 0 : \forall x_0,\ \forall t > 0,\ \|x(t)\|_{\mathcal{H}} \leq C\,e^{-\omega t}$

Strategy:

1. Find dynamical system in state-space Φ to compute $z \star \boldsymbol{u}\cdot\boldsymbol{n}$
2. Formulate an extended Cauchy problem
   $\dot{X} = \mathcal{A}X$, with extended state $X = (\boldsymbol{u}, p, \varphi) \in L^2(\Omega)^{n+1} \times L^2(\Gamma; \Phi)$.
3. Study energy balance: $\dot{\mathcal{E}} \leq 0$, use Lümer-Phillips (Pazy 1983, Thm. 4.3).
4. Inspect $\sigma(\mathcal{A})$, if needed for stability.

Well-posedness: memoryless damping

Kernel. Pure resistance $z(t) = a_0\, \delta_0(t)$ with $a_0 > 0$.

Functional setup. $\mathcal{H} = (L^2(\Omega))^n \times L^2(\Omega)$, $V = (H^1(\Omega))^n \times H^1(\Omega)$

$$\mathcal{A}_{ac} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} -\nabla p \\ -\nabla \cdot \boldsymbol{u} \end{bmatrix},\ \mathcal{D}(\mathcal{A}_{ac}) = \left\{ (\boldsymbol{u}, p) \in V \mid p_{|\Gamma} = a_0 \boldsymbol{u} \cdot \boldsymbol{n}_{|\Gamma} \right\}.$$

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Application of Lümer-Phillips. (Luo, Guo and Morgül 2012, 2.29)

- "$\mathcal{A}_{ac}$ is dissipative"
$$\dot{\mathcal{E}}(t) = (\mathcal{A}_{ac} X, X)_{\mathcal{H}} = -\int_\Gamma p\, \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = -a_0 \|\boldsymbol{u} \cdot \boldsymbol{n}\|^2_{L^2(\Gamma)} \leq 0.$$
- "$\exists \lambda > 0 : \lambda I - \mathcal{A}_{ac}$ surjective" (see (Haddar and Matignon 2008, INRIA))
  - Weak formulation: find $p \in H^1(\Omega)$ such that $\forall \theta \in H^1(\Omega)$
  $$(\nabla p, \nabla \theta)_{L^2(\Omega)} + \frac{\lambda}{a_0} (p, \theta)_{L^2(\Gamma)} + \lambda^2 (p, \theta)_{L^2(\Omega)} = (l, \theta)_{H^1(\Omega)}.$$
  - Then $\exists ! \boldsymbol{u} \in H^1(\mathrm{div}; \Omega)$.
  - Regularity: $\boldsymbol{u} \cdot \boldsymbol{n} = {}^{p}/_{a_0}$ in $H^{-1/2}(\Gamma) \Rightarrow \boldsymbol{u} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma) \Rightarrow$ $\boldsymbol{u} \in H^1(\Omega)$, assuming Ω Lipschitz (Costabel 1990, MMAS).

$\Rightarrow \mathcal{A}_{ac}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. □
This proof *should* not break down provided that z is "dissipative".

Well-posedness: memoryless & long-memory damping

Kernel. $z(t) = a_0\, \delta_0(t) + a_{1/2} \frac{\mathbb{1}_{(0,\infty)}(t)}{\sqrt{\pi t}}$ with $a_0, a_{1/2} > 0$.

Parabolic realisation. $\hat{z}(s)$ irrational $\Rightarrow$ ∞-dimensional realisation.

(Curtain and Zwart 1995)

$$\frac{1}{\sqrt{\pi t}} = \int_0^\infty e^{-\xi t}\, \mathrm{d}\mu(\xi)$$

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$$\frac{1}{\sqrt{\pi t}} = \int_0^\infty e^{-\xi t}\,\mathrm{d}\mu(\xi) \Rightarrow \begin{cases} \partial_t \varphi(t,\xi) = -\xi\,\varphi(t,\xi) + \boldsymbol{u}(t)\cdot\boldsymbol{n} \\ p(t) = a_0\,\boldsymbol{u}(t)\cdot\boldsymbol{n} + a_{1/2}\int_0^\infty \varphi(t,\xi)\,\mathrm{d}\mu(\xi) \end{cases}$$

- State variable $\varphi(t,\cdot) \in \Phi \coloneqq L^2(0,\infty;\mathrm{d}\mu)$ with $\mathrm{d}\mu = \frac{1}{\sqrt{\pi\xi}}\mathrm{d}\xi$.
  (Hélie and Matignon 2006a, M3AS) (Matignon 2013)

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Energy balance.

$$\overbrace{p\boldsymbol{u}\cdot\boldsymbol{n}(t)}^{\text{supplied power}} = \overbrace{\frac{a_{1/2}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_\Phi^2}^{\text{internal power}} + \overbrace{a_0\,|\boldsymbol{u}\cdot\boldsymbol{n}|^2 + a_{1/2}\,\|\sqrt{\xi}\,\varphi\|_\Phi^2}^{\text{memoryless and long-memory dissipation}} \;.$$

$$\geq \frac{a_{1/2}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_\Phi^2$$

Hence, the proof *should* extend to this case!

Well-posedness: memoryless & long-memory damping (cont.)

(Partial) Functional setup. $\mathcal{H} = (L^2(\Omega))^n \times (L^2(\Omega)) \times L^2(\Gamma;\Phi)$.

$$\mathcal{A}\begin{pmatrix} \boldsymbol{u} \\ p \\ \varphi \end{pmatrix} = \begin{pmatrix} -\nabla p \\ -\nabla\cdot\boldsymbol{u} \\ -\xi\,\varphi + \boldsymbol{u}\cdot\boldsymbol{n} \end{pmatrix}.$$

(Partial) Application of Lümer-Phillips.

1. "$\mathcal{A}$ is dissipative"

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|(\boldsymbol{u},p)\|_2^2 + \int_\Gamma a_{1/2}\,\|\varphi\|_\Phi^2\,\mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x})\right] = (\mathcal{A}X, X)_{\mathcal{H}}$$

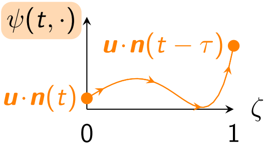
$$\begin{aligned}(\mathcal{A}X, X)_{\mathcal{H}} &= -\int_\Gamma \left[p - a_{1/2}\int_0^\infty \varphi\,\mathrm{d}\mu(\xi)\right]\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} - \left\|\sqrt{a_{1/2}\xi}\,\varphi\right\|^2_{L^2(\Gamma;\Phi)} \\ &= -\int_\Gamma a_0\,|\boldsymbol{u}\cdot\boldsymbol{n}|^2\,\mathrm{d}\boldsymbol{\sigma} - \left\|\sqrt{a_{1/2}\xi}\,\varphi\right\|^2_{L^2(\Gamma;\Phi)} \\ &\leq 0.\end{aligned}$$

$\Rightarrow\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.

Well-posedness: memoryless damping & delay

Kernel. $z(t) = a_0\,\delta_0(t) + a_\tau\,\delta_{-\tau}(t)$ with $a_0, a_\tau, \tau > 0$.

Hyperbolic realisation. $\hat{z}(s)$ irrational $\Rightarrow$ ∞-dimensional realisation.



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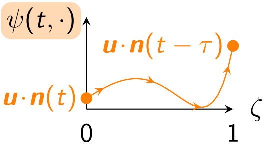

$$\Rightarrow \begin{cases} \partial_t \psi(t,\zeta) = -\dfrac{1}{\tau}\partial_\zeta \psi(t,\zeta), \text{ with } \psi(t,0) = \boldsymbol{u}\cdot\boldsymbol{n} \\ p(t) = a_0\,\boldsymbol{u}(t)\cdot\boldsymbol{n} + a_\tau\,\psi(t,1) \end{cases}$$

- State variable $\psi(t,\cdot) \in \Psi := L^2(0,1;\mathbb{C})$.
  (Curtain and Zwart 1995, § 2.4) (Engel and Nagel 2000, § VI.6)

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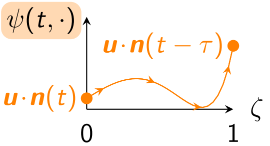

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- State variable $\psi(t,\cdot) \in \Psi := L^2(0,1;\mathbb{C})$.
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Energy balance. No dissipation. There is no "$p\,\boldsymbol{u}\cdot\boldsymbol{n}(t)$".

$$\overbrace{\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_\Psi^2}^{\text{internal power}} = \frac{1}{2}\Big[\overbrace{|\boldsymbol{u}(t)\cdot\boldsymbol{n}|^2}^{\text{input}} - \overbrace{|\boldsymbol{u}(t-\tau)\cdot\boldsymbol{n}|^2}^{\text{output}}\Big].$$

(Partial) Functional setup. $\mathcal{H} = (L^2(\Omega))^n \times (L^2(\Omega)) \times L^2(\Gamma;\Psi)$

(Partial) Lümer-Phillips. Applies provided that
(Monteghetti, Haine and Matignon 2017, IFAC WC)

$$\Re[a_0] > |a_\tau|.$$

Asymptotic stability

Summary. W.P. for $\hat{z}(s) = a_0 + a_{1/2} \, {}^{1}\!/\!\sqrt{s} + a_\tau e^{-s\tau}$ using realisations.

Parabolic realisation (ODE)	Hyperbolic realisation (PDE)
$\xi \in (0, \infty)$	$\zeta \in (0, 1)$

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Stability. If $\hat{z}(s)$ rational $\Rightarrow$ energy method is enough. (LaSalle's I.P.).

If $\hat{z}(s)$ irrational $\Rightarrow \mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ may not compact $\Rightarrow$ inspect $\sigma(\mathcal{A})$.

Asymptotic stability theorem (Arendt and Batty 1988) (Lyubich and Vũ 1988)

Let $e^{t\mathcal{A}}$ be a uniformly bounded C_0-semigroup on $\mathcal{H}$. If

- $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_c(\mathcal{A})$
- $\sigma_c(\mathcal{A})$ countable

$\Rightarrow$ $e^{t\mathcal{A}}$ is asymptotically stable.

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Application for long-memory and delay.
Show that $i\mathbb{R}^* \subset \rho(A)$ (see (Matignon and Prieur 2014, MCRF)).
$\mathcal{A}$ is closed, so (Curtain and Zwart 1995, Def. A.4.4)

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \,|\, \mathcal{N}(\mathcal{A}_\lambda) = \{0\} \text{ and } R(\mathcal{A}_\lambda) = \mathcal{H}\}.$$

- Fredholm alternative on weak formulation, using embedding $H^1(\Omega) \subset\subset H^{1/2}(\Omega)$ (Lions and Magenes 1972, Thm. 16.1).
- $0 \notin \sigma_p(\mathcal{A})$. □

Outline

Discontinuous Galerkin method (DG)

Linearised Euler equations on $\Omega \subset \mathbb{R}^n$ with base flow $\boldsymbol{u}_0$

$$\partial_t \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} + \begin{bmatrix} (\boldsymbol{u_0} \cdot \nabla)\boldsymbol{u} + \nabla p \\ \nabla \cdot \boldsymbol{u} + \boldsymbol{u_0} \cdot \nabla p \end{bmatrix} + \begin{bmatrix} (\boldsymbol{u} \cdot \nabla)\boldsymbol{u_0} + p\,(\boldsymbol{u_0} \cdot \nabla)\boldsymbol{u_0} \\ p\,\nabla \cdot \boldsymbol{u_0} \end{bmatrix} = \boldsymbol{0}$$

Weak formulation. $\boldsymbol{q} := [\boldsymbol{u}, p] \in V$, $\boldsymbol{\theta} := (\boldsymbol{\theta^u}, \theta^p) \in \mathcal{C}^\infty(\overline{\Omega})^{n+1}$

$$(\partial_t \boldsymbol{q}, \boldsymbol{\theta})_\Omega - (A_i \cdot \boldsymbol{q}, \partial_i \boldsymbol{\theta})_\Omega + (B \cdot \boldsymbol{q}, \boldsymbol{\theta})_\Omega = -\int_{\partial\Omega} \begin{bmatrix} \boldsymbol{n} \cdot f_p \;\boldsymbol{\theta^u} \\ \boldsymbol{n} \cdot \boldsymbol{f_u} \;\theta^p \end{bmatrix} \mathrm{d}\boldsymbol{\sigma}$$

DG formulation. Triangulation Ω_h. $\boldsymbol{q_h} \in V_h$, $\boldsymbol{\theta_h} \in V_h$

$$(\partial_t \boldsymbol{q_h}, \boldsymbol{\theta_h})_{\Omega_k} - (A_i \cdot \boldsymbol{q_h}, \partial_i \boldsymbol{\theta_h})_{\Omega_k} + (B \cdot \boldsymbol{q_h}, \boldsymbol{\theta_h})_{\Omega_k} = -\int_{\partial\Omega_k} \begin{bmatrix} \boldsymbol{n} \cdot f_p^* \;\boldsymbol{\theta_h^u} \\ \boldsymbol{n} \cdot \boldsymbol{f_u^*} \;\theta_h^p \end{bmatrix} \mathrm{d}\boldsymbol{\sigma}$$

DG formulation features

- V_h: Lagrange basis, size N_p (Hesthaven and Warburton 2008, § 6.1)
- Upwind flux for $\boldsymbol{u_0} \in \mathcal{C}^1(\overline{\Omega})^n$ (Hesthaven and Warburton 2008, § 2.4)

$\Rightarrow$ Impedance boundary condition?

Impedance boundary condition

Impedance boundary condition.

$$p(x,t) = a_0\, \boldsymbol{u}\cdot\boldsymbol{n}(x,t) + a_i\, Q_i(\boldsymbol{u}(x,t)\cdot\boldsymbol{n}) \quad x \in \Gamma$$

where $a_0, a_i \geq 0$ and $Q_i \neq I$ is

- A delay: $Q_i(\boldsymbol{u}\cdot\boldsymbol{n}) = \boldsymbol{u}(\cdot - \tau)\cdot\boldsymbol{n}$, or
- An operator with dissipative realisation in state-space Φ_i. Examples: "∂_t" ($\Phi_i = \mathbb{R}$) and "$\partial_t^{1/2}$" ($\Phi_i = L^2(0,\infty; \mathrm{d}\mu)$).

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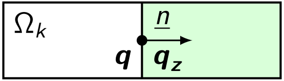
Assumption: $\boldsymbol{u_0}\cdot\boldsymbol{n} = 0$ on Γ, so that

$$\boldsymbol{f}_{\boldsymbol{q}} = \boldsymbol{q} = [\boldsymbol{u}, p].$$

Numerical flux function: central flux with ghost state

$$\boldsymbol{f}^*_{\boldsymbol{q}}(\boldsymbol{q}) := \frac{1}{2}(\boldsymbol{q} + \boldsymbol{q}_z)$$



Ghost state expression

Theorem. L^2-stability.

The general expression of the ghost state $\boldsymbol{q_z}$ is

$$\boldsymbol{q_z} = \begin{bmatrix} -\alpha\, I & \frac{1}{a_0}(1+\alpha)\,\boldsymbol{n} \\ a_0(1-\alpha)\,\boldsymbol{n}^\mathsf{T} & \alpha \end{bmatrix} \cdot \boldsymbol{q} + a_i \begin{bmatrix} -\frac{1}{a_0}(1+\alpha)\,\boldsymbol{n} \\ 1-\alpha \end{bmatrix} Q_i(\boldsymbol{u}\cdot\boldsymbol{n})$$

- If $a_i = 0$, then L^2-stability is achieved for $\alpha \in [-1, 1]$.
- If $a_i \neq 0$, then L^2-stability is achieved for $\alpha = -1$.

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Noticeable values:

- $\alpha = 1 \iff p_z = p$, $\alpha = -1 \iff \boldsymbol{u_z} = \boldsymbol{u}$
- $\alpha = \beta_0$, with $\beta_0 = (a_0 - 1)/(a_0 + 1)$ (reflection coefficient)

$$\boldsymbol{q_z} = \begin{bmatrix} -1\, I & \frac{2}{a_0}\,\boldsymbol{n} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix} \cdot \boldsymbol{q} + a_i \begin{bmatrix} -\frac{2}{a_0}\,\boldsymbol{n} \\ 0 \end{bmatrix} Q_i(\boldsymbol{u}\cdot\boldsymbol{n})$$

Ghost state expression

Theorem. L^2-stability.

The general expression of the ghost state $\boldsymbol{q_z}$ is

$$\boldsymbol{q_z} = \begin{bmatrix} -\alpha\, \boldsymbol{I} & \frac{1}{a_0}(1+\alpha)\,\boldsymbol{n} \\ a_0(1-\alpha)\,\boldsymbol{n}^\intercal & \alpha \end{bmatrix} \cdot \boldsymbol{q} + a_i \begin{bmatrix} -\frac{1}{a_0}(1+\alpha)\,\boldsymbol{n} \\ 1-\alpha \end{bmatrix} Q_i(\boldsymbol{u}\cdot\boldsymbol{n})$$

- If $a_i = 0$, then L^2-stability is achieved for $\alpha \in [-1, 1]$.
- If $a_i \neq 0$, then L^2-stability is achieved for $\alpha = -1$.

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$$\boldsymbol{q_z} = \begin{bmatrix} 1\,\boldsymbol{I} & \boldsymbol{0} \\ 2a_0\,\boldsymbol{n}^\intercal & -1 \end{bmatrix} \cdot \boldsymbol{q} + a_i \begin{bmatrix} \boldsymbol{0} \\ 2 \end{bmatrix} Q_i(\boldsymbol{u}\cdot\boldsymbol{n})$$

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Proof. Let us define

$$\boldsymbol{q}_z := \begin{bmatrix} \alpha_1\, I & \alpha_2\, \boldsymbol{n} \\ \alpha_3\, \boldsymbol{n}^\mathsf{T} & \alpha_4 \end{bmatrix} \cdot \boldsymbol{q} + \begin{bmatrix} \gamma_1^i \\ \gamma_2^i \end{bmatrix} Q_i(\boldsymbol{u}\cdot\boldsymbol{n}).$$

The proof breaks down into two steps:

1. Compatibility conditions
2. Discrete energy balance

Ghost state expression (2/3)

Generic expression of $\boldsymbol{q_z}$

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1. Compatibility conditions.
   (a) For $\boldsymbol{q} = \boldsymbol{q}_z$, (1) should give

   $$p = a_0 \boldsymbol{u} \cdot \boldsymbol{n} + a_i Q_i(\boldsymbol{u}\cdot\boldsymbol{n}).$$

   (b) Since we use a central flux, (1) should give

   $$\frac{p + p_z}{2} = a_0 \frac{\boldsymbol{u} + \boldsymbol{u}_z}{2} \cdot \boldsymbol{n} + a_i Q_i(\boldsymbol{u}\cdot\boldsymbol{n}).$$

Both these conditions yield

$$\boldsymbol{q_z} = \begin{bmatrix} -\alpha \, I & \frac{1}{a_0}(1+\alpha) \, \boldsymbol{n} \\ a_0(1-\alpha) \, \boldsymbol{n}^\mathsf{T} & \alpha \end{bmatrix} \cdot \boldsymbol{q} + a_i \begin{bmatrix} -\frac{1}{a_0}(1+\alpha) \, \boldsymbol{n} \\ 1-\alpha \end{bmatrix} Q_i(\boldsymbol{u}\cdot\boldsymbol{n}),$$

where α is a **seemingly** free parameter.

Ghost state expression (3/3)

Let us assume that Q_i has a dissipative realisation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi^i\|^2_{\Phi_i} \leq a_i Q_i(\boldsymbol{u}\cdot\boldsymbol{n})\,\boldsymbol{u}\cdot\boldsymbol{n},$$

so that the continuous energy balance is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\boldsymbol{u}\|^2_2 + \|p\|^2_2 + \|\sqrt{a_i}\varphi^i\|^2_{L^2(\Gamma;\Phi_i)}\right] \leq -\int_{\partial\Omega} \overbrace{\left[p - a_i Q_i(\boldsymbol{u}\cdot\boldsymbol{n})\right]\boldsymbol{u}\cdot\boldsymbol{n}}^{=a_0|\boldsymbol{u}\cdot\boldsymbol{n}|^2}\,\mathrm{d}\boldsymbol{\sigma}.$$

2. Discrete energy balance on element k (w/o upwind contrib.)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k_h \leq -\int_{\partial\Omega_k}\left[\frac{p_z\boldsymbol{u_h} + p_h\boldsymbol{u_z}}{2}\cdot\boldsymbol{n} - a_i Q_i(\boldsymbol{u_h}\cdot\boldsymbol{n})\right]\mathrm{d}\boldsymbol{\sigma}$$

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$$\leq -\frac{1}{2}\int_{\partial\Omega_k}\left[\frac{1+\alpha}{a_0}|p_h|^2 + (1-\alpha)a_0|\boldsymbol{u_h}\cdot\boldsymbol{n}|^2\right]\mathrm{d}\boldsymbol{\sigma} + \frac{1}{2}\int_{\partial\Omega}\Delta_i Q_i(\boldsymbol{u_h}\cdot\boldsymbol{n})\mathrm{d}\boldsymbol{\sigma},$$

where $\Delta_i := a_i(1+\alpha)\left[\frac{1}{a_0}p_h + \boldsymbol{u_h}\cdot\boldsymbol{n}\right]$.

Distinguish the cases $a_i = 0$ and $a_i \neq 0$ to conclude.
(Delay case similar.) □

Summary

Summary. For IBC $p = a_0 \boldsymbol{u}\cdot\boldsymbol{n} + a_i Q_i(\boldsymbol{u}\cdot\boldsymbol{n})$, the DG formulation is

$$(\partial_t \boldsymbol{q_h}, \boldsymbol{\theta_h})_{\Omega_k} = \overbrace{\cdots}^{\text{standard}} - \int_{\partial\Omega_k} a_i \begin{bmatrix} \frac{1}{a_0}(1+\alpha)\,\theta_h^p \\ (1-\alpha)\boldsymbol{n}\cdot\boldsymbol{\theta_h^u} \end{bmatrix} Q_i(\boldsymbol{u_h}\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{\sigma}.$$

Accurate evaluation of Q_i $\Rightarrow$ high-order simulation.

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Accurate evaluation of Q_i $\Rightarrow$ high-order simulation.

Example. $\hat{Q}_1(s) = 1/\sqrt{s}$, $\hat{Q}_2(s) = e^{-\tau s}$

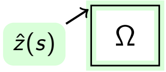
Parabolic realisation (ODE)	Hyperbolic realisation (PDE)
$\xi \in (0,\infty)$	$\zeta \in (0,1)$
$\partial_t \varphi_h = -\xi\varphi_h + \boldsymbol{u_h}\cdot\boldsymbol{n}$	$\partial_t \psi_h = -\tau^{-1}\partial_\zeta \psi_h$
$Q_1(\boldsymbol{u_h}\cdot\boldsymbol{n}) = \sum_{k\in[\![1,N_\xi]\!]} \mu_k\, \varphi_h(\xi_k)$	$\psi_h(0) = \boldsymbol{u_h}\cdot\boldsymbol{n}$
	$Q_2(\boldsymbol{u_h}\cdot\boldsymbol{n}) = \psi_h(\zeta = 1)$
$\Rightarrow$ Quadrature or optimisation	$\Rightarrow$ Monodimensional DG

Global (time-local) formulation:

$$\dot{\boldsymbol{x}}_{\boldsymbol{h}} = \mathcal{A}_h \boldsymbol{x_h} \quad \text{with } \boldsymbol{x_h} := (\boldsymbol{q_h}, \boldsymbol{\varphi_h}, \boldsymbol{\psi_h}).$$

Stability study $\Rightarrow \omega_0(\mathcal{A}_h)$ (approximation of $\omega_0(\mathcal{A})$).

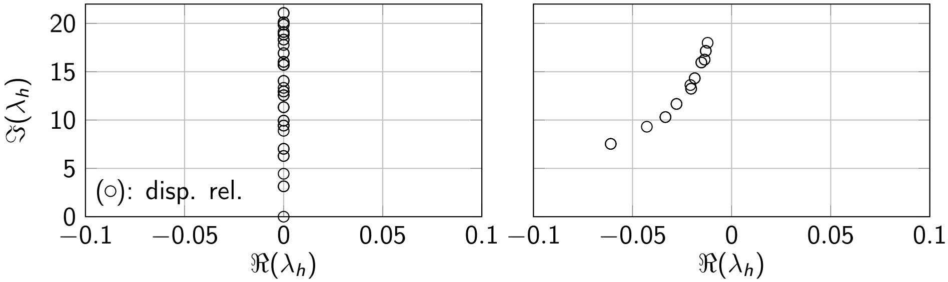
Illustration: acoustical cavity



$\Omega = (0,1)^2$, no base flow $\boldsymbol{u_0} = \boldsymbol{0} \Rightarrow \dot{\boldsymbol{x}}_{\boldsymbol{h}} = \mathcal{A}_h \boldsymbol{x}_{\boldsymbol{h}}$.

1. Impact of α on growth rate $\omega_0(\mathcal{A}_h)$. $\hat{z}(s) = a_0 + a_1 s$

Rigid wall $a_0 = a_1 = 10^{10}$ Soft wall $a_0 = a_1 = 1$



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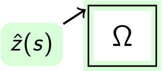

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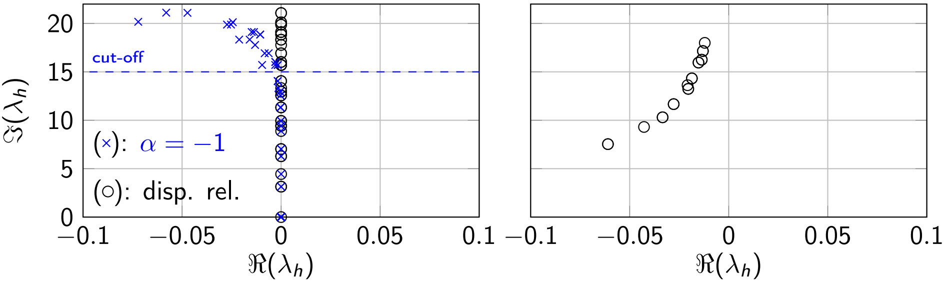

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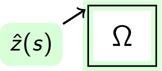

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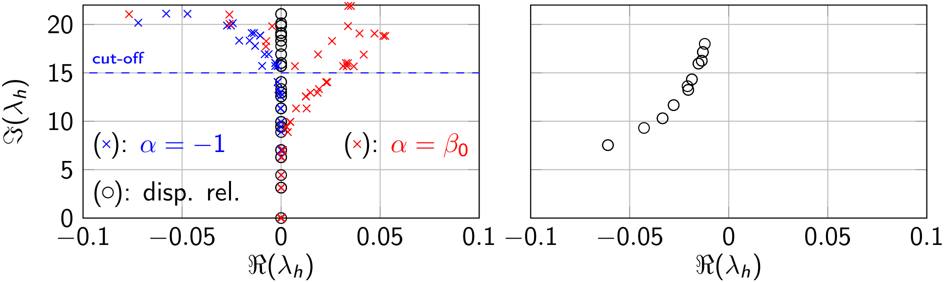

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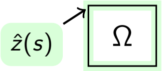

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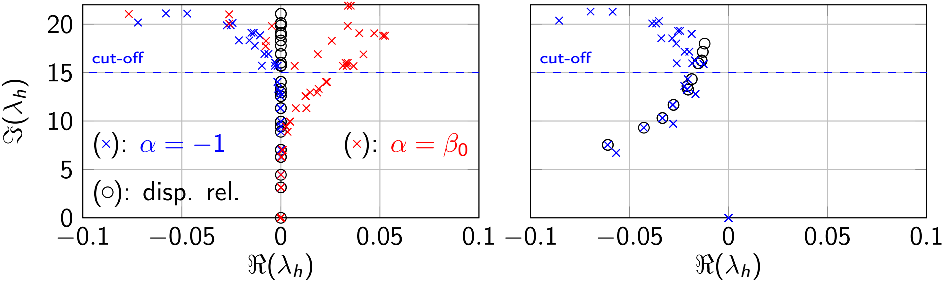

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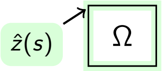

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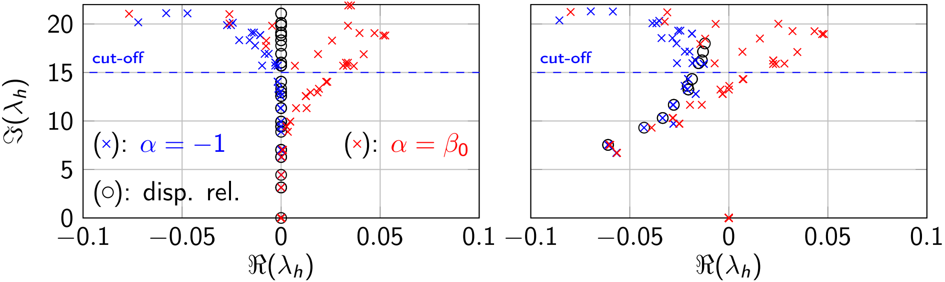

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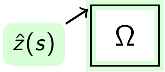

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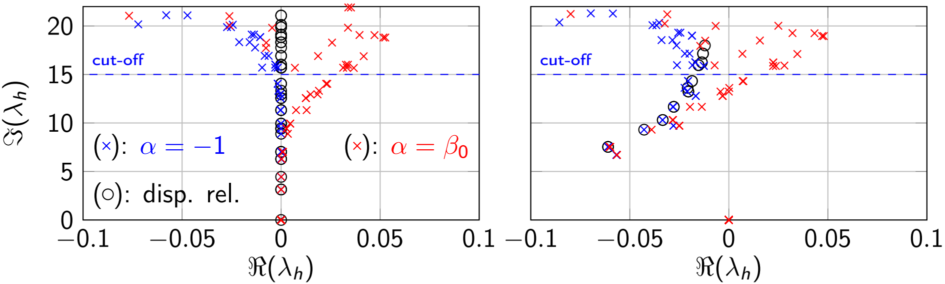


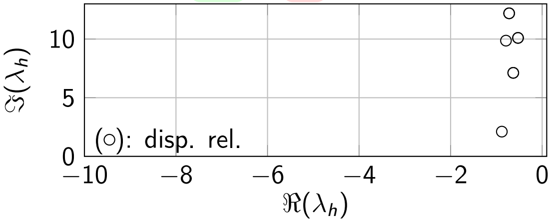
$\Rightarrow$ At the other end of the spectrum...

- Value $\alpha = -1$ leads to scaling $\max_{\lambda_h \in \sigma(\mathcal{A}_h)} |\lambda_h| \underset{a_0 \to \infty}{=} \mathcal{O}(a_0)$.
- Value $\alpha = \beta_0 \in [-1, 1]$ yields $\mathcal{O}(1)$.

Illustration: acoustical cavity (cont.)

2. $\partial_t^{1/2}$ ⇒ Essential spectrum

$\hat{z}(s) = \sqrt{s}$ $N_\xi = 100$



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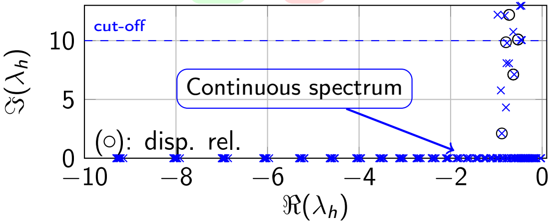

Illustration: acoustical cavity (cont.)

2. $\partial_t^{1/2} \Rightarrow$ Essential spectrum

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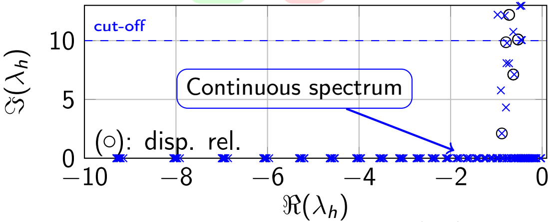

3. Stability condition with time-delay: $a_0 > |a_\tau|$

$$\hat{z}(s) = a_0 + e^{-100s},\ N_p = 5,\ N_k = 1$$

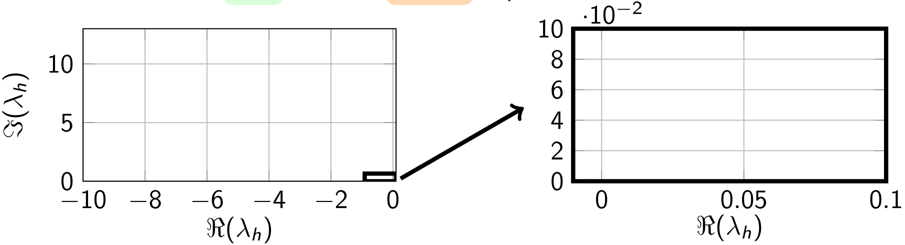

Illustration: acoustical cavity (cont.)

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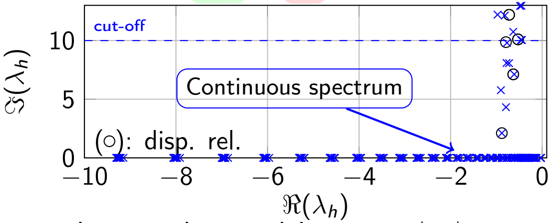

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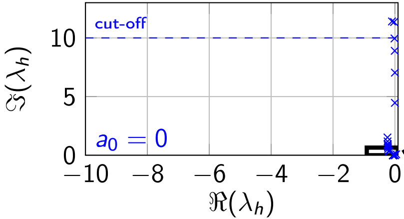

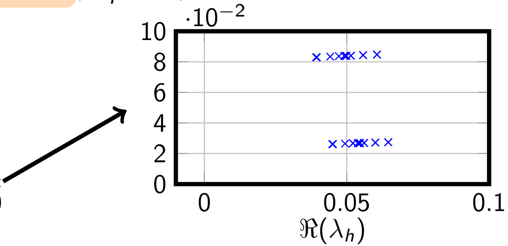

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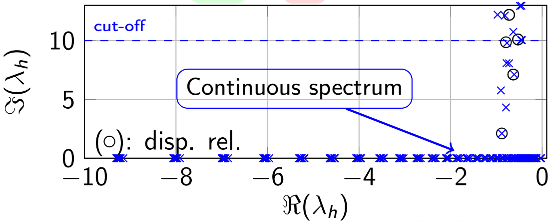

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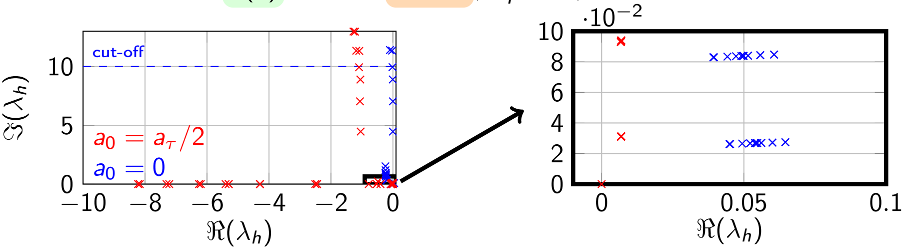

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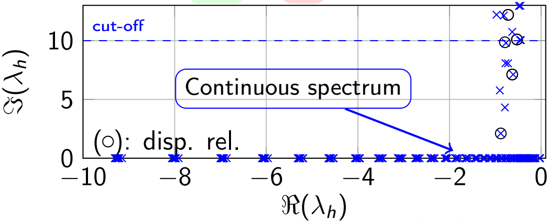

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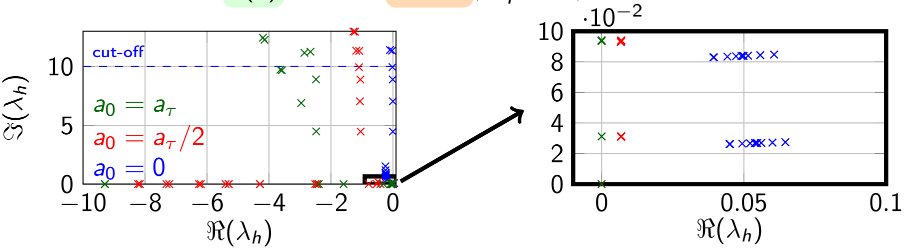
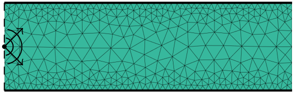


Illustration: time-domain simulation

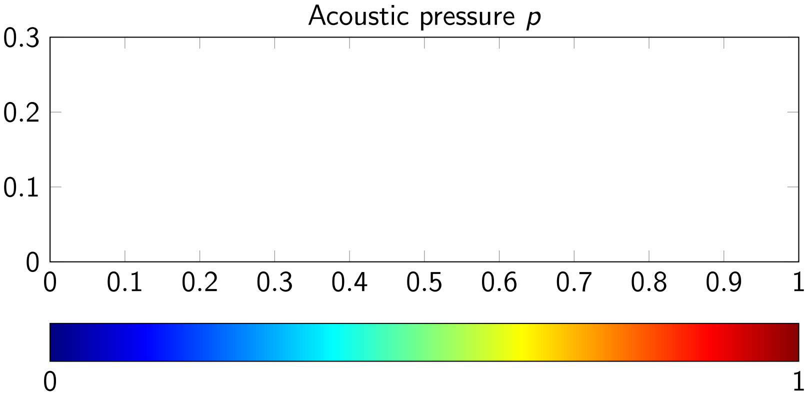
Computational case Infinite 2D duct.

DG: $N = 4$. Mesh: $N_K = 688$.

Time-integration: CFL $= 0.5$. (LSERK (8,4) (Toulorge and Desmet 2012))



$\hat{z}(s, x) = \infty$ (Rigid Wall)



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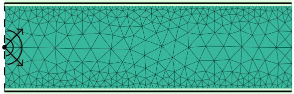
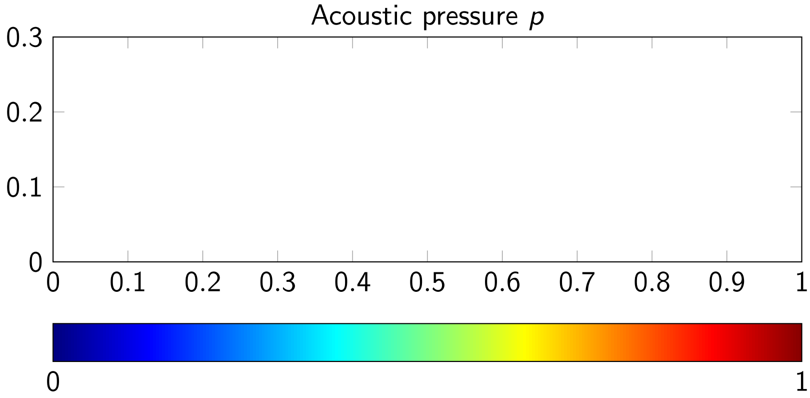


$$\hat{z}(s, x) = a_{1/2}\sqrt{s} \text{ (Soft Wall)}$$



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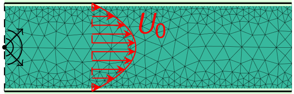

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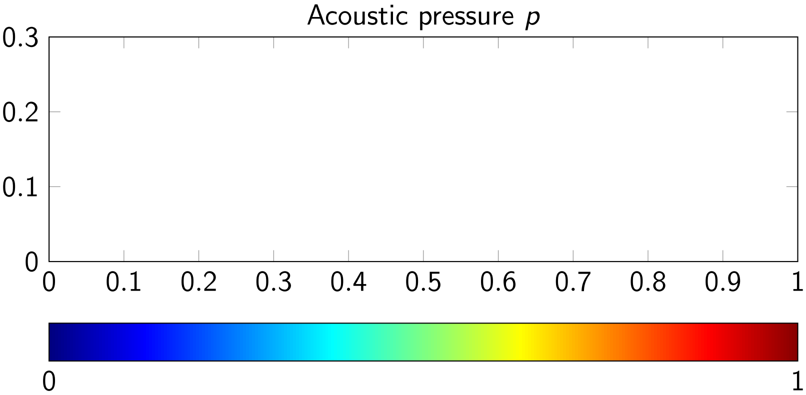
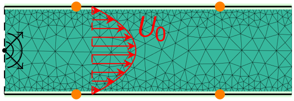


Illustration: time-domain simulation

Computational case Infinite 2D duct.

DG: $N = 4$. Mesh: $N_K = 688$.

Time-integration: CFL $= 0.5$. (LSERK (8,4) (Toulorge and Desmet 2012))



$$\hat{z}(s,x) = a_{1/2}(x)\sqrt{s} \text{ (Soft Wall)}$$

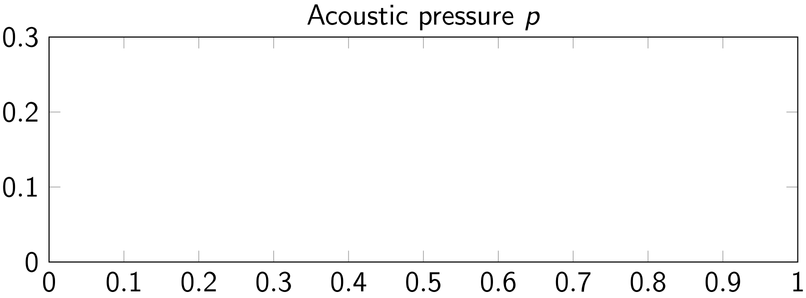


Illustration: time-domain simulation

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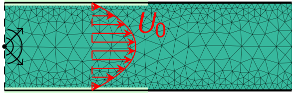
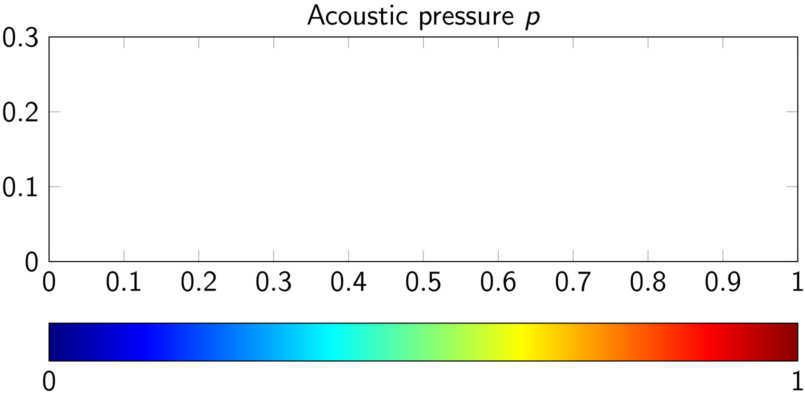


$\hat{z}(s, x) = 0$ (Soft-Hard transition)



Outline

Conclusion

Takeaways

- Time-local formulation (parabolic φ / hyperbolic ψ) of realistic impedance model $\hat{z}(s) = 1/\sqrt{s} + e^{-s\tau}$
- W.P. & stability (coupled formulation $(p, \boldsymbol{u}, \varphi, \psi)$)
- Stable DG formulation (ghost state $\boldsymbol{q_z}$) with dissipative boundary operator Q

  $\Rightarrow$ High-order time-domain simulation (CAA)
  $\Rightarrow$ Eigenvalue approach to stability

Perspectives

- Proof extensions
- Control problems
- Advanced applications
- Electromagnetics (CEM)

Conclusion

1. Introduction
2. Acoustical case: Theory
3. Aeroacoustical case: Numerical method
4. Conclusion

▸ Appendix

Thanks for your attention. Any questions?

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