Computation of plasmon resonances localized at corners using frequency-dependent complex scaling

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Abstract

A plasmonic device with a non-smooth boundary can exhibit strongly-oscillating surface waves whose phase velocities vanish as they reach the corners. This work investigates in the quasi-static limit the existence of corner resonances, which are analogous to scattering resonances in the sense that the local behavior at each corner plays the role of the behavior at infinity. Resonant contrasts are sought as eigenvalues of the transmission problem with complex scaling applied at corners. Since the scaling function must depend upon the contrast, the corresponding eigenvalue problem is nonlinear.

Keywords: sign-changing permittivity, complex resonances, complex scaling

1 Definition of corner resonances

This work focuses on the transmission problem

$$\begin{cases} \operatorname{div}\left(\varepsilon(\boldsymbol{x})^{-1}\nabla u(\boldsymbol{x})\right) = 0 & \left(\boldsymbol{x}\in\Omega\subset\mathbb{R}^2\right) \\ u(\boldsymbol{x}) = 0 & \left(\boldsymbol{x}\in\partial\Omega\right), \end{cases} \tag{1}$$

where the differentiations are weak, Ω is a bounded $\mathcal{C}^\infty$ domain, and ε is piecewise constant

$$\varepsilon(\boldsymbol{x}) = \varepsilon_m \mathbb{1}_{\Omega_m}(\boldsymbol{x}) + \varepsilon_d \mathbb{1}_{\Omega\setminus\overline{\Omega_m}}(\boldsymbol{x}), \tag{2}$$

where Ω_m is a piecewise-smooth domain modeling the plasmonic device, see Figure 1. Typically, ε_m depends upon the frequency ω through a physical model; this dependency need not be introduced herein since ω does not appear explicitly in the quasi-static approximation (1). If the *contrast*

$$\kappa := \frac{\varepsilon_m}{\varepsilon_d}$$

is not real, then the only solution of (1) in $H^1(\Omega)$ is $u = 0$. Let us review some results for $\kappa\in\mathbb{R}$.

If $\partial\Omega_m$ is smooth there is a sequence of *real* eigenvalues $(\kappa_n)_n$, for which (1) has a non-null solution, accumulating at -1 [3, Thm. 1]. These contrasts are associated with surface waves known as *surface plasmons,* whose energy-concentrating properties are employed in many applications.

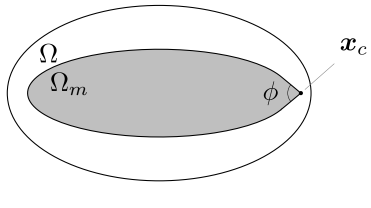

Figure 1: Transmission problem on Ω. The plasmonic inclusion Ω_m has a corner of angle ϕ.

Let $\partial\Omega_m$ have one corner $\boldsymbol{x}_c$ of angle $\phi\in(0,\pi)$. If κ lies in the *critical interval* [1, Tab. 1] [4]

$$I_c := \left[\frac{\phi-2\pi}{\phi}, -1\right] \cup \left[-1, \frac{\phi}{\phi-2\pi}\right] = I_c^{\text{odd}} \cup I_c^{\text{even}},$$

then there is $\eta\in\mathbb{R}$, which depends implicitly upon the contrast κ through the dispersion relation

$$f_\phi(\eta,\kappa) := \left[\frac{\sinh(\eta\pi)}{\sinh\left[\eta\left(\pi-\phi\right)\right]}\right]^2 - \left[\frac{1-\kappa}{1+\kappa}\right]^2 = 0, \tag{3}$$

such that the strongly-oscillating *black-hole* field

$$u_{\text{bh}}(r,\theta) = r^{i\eta}\Phi(\theta) \quad (\theta\in(-\pi,\pi])$$

is a solution to (1) in a neighborhood of $\boldsymbol{x}_c$. Note that $u_{\text{bh}}\in L^2(\Omega)$ but $u_{\text{bh}}\notin H^1(\Omega)$. The requirement that u_{bh} be outgoing leads to (using energy considerations or a limiting absorption principle)

$$\eta < 0 \text{ if } \kappa\in I_c^{\text{odd}},\ \eta > 0 \text{ if } \kappa\in I_c^{\text{even}}. \tag{4}$$

The purpose of this work is to investigate the existence of complex resonances occurring at the corners of $\partial\Omega_m$, which can be built analogously to that of usual scattering resonances. We propose here the following, somewhat imprecise, definition:

Definition 1. A *corner resonance* is solution to (1) that is outgoing at a corner $\boldsymbol{x}_c\in\partial\Omega_m$.

A corner resonance is localized at a corner $\boldsymbol{x}_c$ of $\partial\Omega_m$ in the sense that it blows up as

$$u(r,\theta) \underset{r\to 0^+}{\sim} r^{i\eta}\Phi(\theta) \quad (\theta\in(-\pi,\pi]), \tag{5}$$

where κ solves (3) with $\Im(\eta)\geq 0$. The multivalued nature of $\kappa\mapsto\{\eta\,|\,f_\phi(\eta,\kappa)=0\}$ suggests that

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such resonances can exist. This seems to be corroborated by the strategy proposed in [5], which consists in perturbing a smooth $\partial\Omega_m$ with corners so that eigenvalues κ_n are perturbed into embedded eigenvalues or resonances.

2 Complex scaling for corners resonances

The principle of complex scaling is to *define* a modified transmission problem such that if (u, κ) is a corner resonance, then κ is an eigenvalue of the modified problem. The construction exploits the analyticity of (5) with respect to r.

To formalize this, let (r, θ) be cylindrical coordinates originating at a corner $\boldsymbol{x}_c \in \partial\Omega_m$. The complex scaling technique introduced in [1] consists in analytically continuing u from $(0, R) \times (-\pi, \pi]$ to $\{r^{1/\alpha} \,|\, r \in (0, R)\} \times (-\pi, \pi]$, where $\alpha \in \mathbb{C}$ is the complex scaling parameter to be *chosen*. By defining $u_\alpha(r, \theta) := u(r^{1/\alpha}, \theta)$, the modified problem is

$$\varepsilon^{-1} \left(\alpha r \partial_r\right)^2 u_\alpha(r, \theta) + \partial_\theta \left(\varepsilon^{-1} \partial_\theta u_\alpha\right)(r, \theta) = 0 \tag{6}$$

on $(0, R) \times (-\pi, \pi]$, which can be discretized with a finite element method. A suitable choice of $\alpha \neq 1$ enables to turn resonant contrasts κ that belong to a given region $K_\alpha \subset \mathbb{C}$ into eigenvalues of (6). The associated eigenfunction $u_\alpha \in L^2(\Omega)$ behaves at each corner as (compare with (5))

$$u_\alpha(r, \theta) \underset{r \to 0^+}{\sim} r^{i\frac{\eta}{\alpha}} \Phi(\theta) \quad (\theta \in (-\pi, \pi]),$$

where $\alpha \in \mathbb{C}$ is chosen such that $\Im(\eta/\alpha) < 0$.

The numerical difficulty stems from the fact that (6) is nonlinear in κ, since $\arg(\alpha)$ *must* depend upon κ, echoing [6] where parameters with frequency-dependent modulus are considered. This is apparent from the outgoing condition (4), which implies

$$\arg(\alpha) < 0 \text{ if } \kappa \in I_c^{\text{odd}}, \ \arg(\alpha) > 0 \text{ if } \kappa \in I_c^{\text{even}}.$$

Specifically, a study of the dispersion relation (3) shows that α must satisfy the stability constraint

$$\theta_{\min}(\kappa) < \arg(\alpha(\kappa)) < \theta_{\max}(\kappa) \quad (\kappa \in \mathbb{R}), \tag{7}$$

which ensures that only outgoing corner waves must be exponentially decaying (equivalently, the corner must not bring energy into the domain).

Figure 2 illustrates (7). The values of $\theta_{\max}$ for $\kappa < \frac{\phi - 2\pi}{\phi}$ and $\theta_{\min}$ for $\kappa > \frac{\phi}{\phi - 2\pi}$ are consequences of (4). Any κ-independent scaling parameter such as $\alpha_1 \equiv e^{-i\pi/4}$ fails (7); $\alpha_2(\kappa) = e^{i\theta(\kappa)}$ where $\theta(\kappa)$ is a polynomial satisfying (7) for $\kappa \in [-6, 0]$.

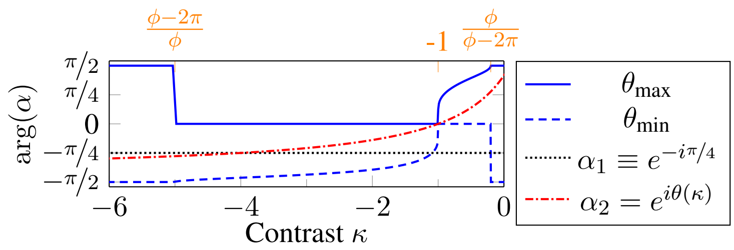

Figure 2: Condition (7) for a corner angle $\phi = \pi/3$.

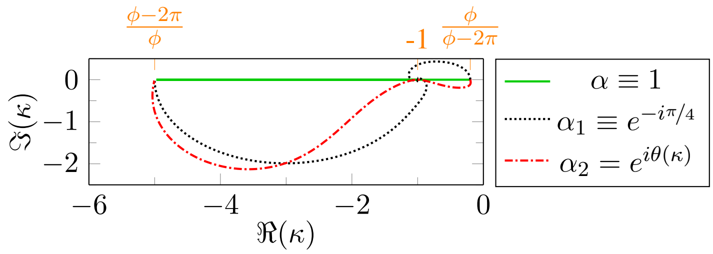

Figure 3: Essential spectrum σ_{ess} for $\phi = \pi/3$.

Figure 3 plots the deformation of the essential spectrum

$$\sigma_{\text{ess}} = \{\kappa \,|\, \exists \eta \in \mathbb{C}^* : \, f_\phi(\eta, \kappa) = 0, \ \Im(\eta/\alpha) = 0\}$$

for $\alpha \equiv 1$ (i.e. no complex scaling, in which case we recover I_c), for the κ-independent scaling α_1 (discussed in [2, § 4.7.1]), and for the κ-dependent scaling α_2. The region K_α where resonant contrasts can be computed is the region uncovered by the deformation of the essential spectrum.

Ongoing work focuses on the construction of a scaling function $\kappa \mapsto \alpha(\kappa)$ that maximizes $|K_\alpha|$ while still leading to a tractable nonlinear eigenvalue problem in κ.

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